UNIVERSITÀ DI PAVIA
FACOLTÀ DI INGEGNERIA
CORSO DI LAUREA IN INGEGNERIA CIVILE E AMBIENTALE/PER L'AMBIENTE E IL TERRITORIO
Esame di Fisica Matematica
22 settembre 2017

Il *candidato* scriva nello spazio sottostante il proprio Cognome e Nome.

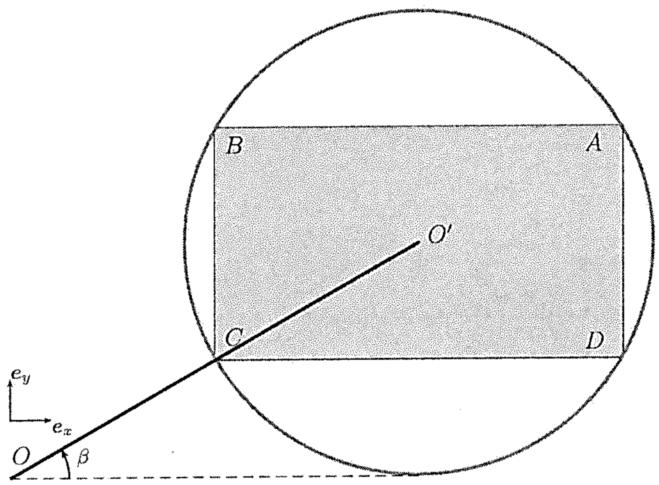
COGNOME **NOME**

La ***prova*** consta di **3** esercizi e durerà **2** ***ore*** e **30** ***minuti***. ***Non è permesso*** usare né calcolatrice né telefono cellulare/smartphone/altri strumenti elettronici; non è consentito consultare testi o appunti, al di fuori di quelli eventualmente distribuiti dalla Commissione pena l'esclusione dalla prova.

Esercizi

1. *Un corpo rigido piano è composto da: un'asta omogenea OO' di massa $2m$ e lunghezza $2R$, un* **anello** *omogeneo di massa $3m$, avente raggio R e centro O', una lamina rettangolare omogenea $ABCD$ con i vertici saldati all'anello, avente massa m e i lati AB di lunghezza $R\sqrt{3}$ e BC di lunghezza R; l'asta passa per il vertice C. Utilizzando la base ortonormale $\{e_x, e_y, e_z\}$ con e_x parallelo ad $A-B$, ed e_y parallelo a $B-C$, determinare:*

1. *gli elementi I_{xx}, I_{yy} e I_{xy} della matrice di inerzia, calcolata rispetto ad O*
   (a) *dell'asta OO' (**3 pt**),*
   (b) *dell'anello (**2 pt**),*
   (c) *della lamina rettangolare $ABCD$ (**3 pt**);*
2. *il momento* **centrale** *di inerzia complessivo per tutto il corpo rispetto alla direzione e_y, precisando i contributi delle tre parti e la loro somma (**4 pt**).*



$$\frac{1}{2} + \frac{3}{2} + \frac{1}{4} + 1 = \frac{2+6+1+4}{4} = \frac{13}{4}$$

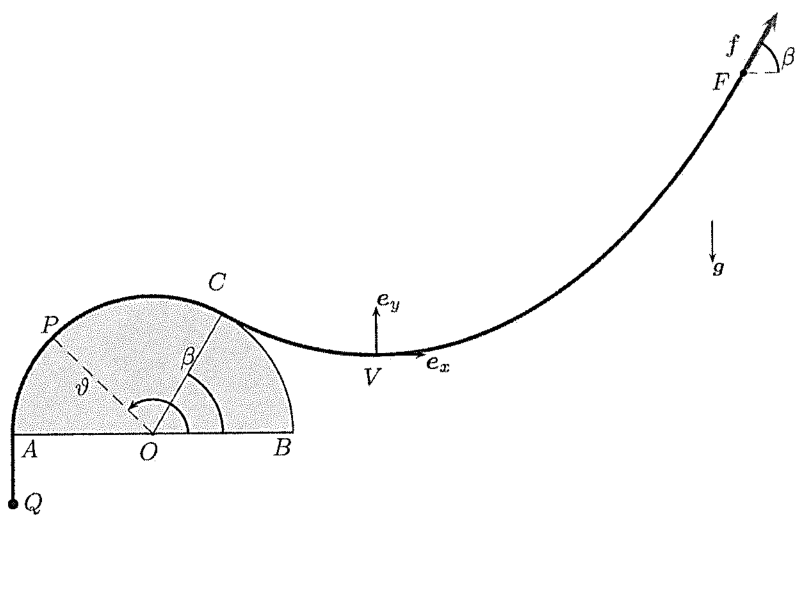
2. *Calcolare per il seguente sistema di vettori applicati:*

$$\begin{cases} \boldsymbol{v}_1 = \boldsymbol{e}_x - 2\boldsymbol{e}_y + 3\boldsymbol{e}_z & \text{applicato in } P_1 - O \equiv (1,1,0), \\ \boldsymbol{v}_2 = \boldsymbol{e}_x + \boldsymbol{e}_y + \boldsymbol{e}_z & \text{applicato in } P_2 - O \equiv (0,-1,1), \\ \boldsymbol{v}_3 = -\boldsymbol{e}_x + \boldsymbol{e}_y + 2\boldsymbol{e}_z & \text{applicato in } P_3 - O \equiv (1,0,-1)\,, \end{cases}$$

1. *il momento risultante del sistema rispetto al polo O* **(2 pt)***;*
2. *il momento risultante del sistema rispetto al polo $Q \equiv (1,1,3)$***(2 pt)***;*
3. *un sistema equipollente di due vettori applicati, di cui uno applicato in Q* **(2 pt)***;*

3. *In un piano verticale, un filo omogeneo QF di peso per unità di lunghezza costante p/R e lunghezza opportuna ha il tratto AQ, di lunghezza $R/2$ libero con un contrappeso $2p$ applicato in Q; il tratto AC appoggiato senza attrito ad un supporto semicircolare fisso di diametro orizzontale $AB = 2R$ e centro O, in modo che il raggio OC formi un angolo $\beta = \pi/3$ con l'orizzontale; infine, una forza $\boldsymbol{f}$ avente modulo γp inclinata di un angolo β rispetto all'orizzontale è applicata in F. In condizioni di equilibrio, determinare:*

1. *la tensione del filo in C* **(2 pt)***;*
2. *la tensione del filo nell'arco AC in funzione dell'angolo $\vartheta = B\hat{O}P$* **(2 pt)***;*
3. *l'equazione dell'arco libero CF, riferito ad assi orizzontali e verticali centrati in V, punto di quota minima dell'arco* **(2 pt)***;*
4. *il valore di γ che garantisce l'equilibrio descritto* **(2 pt)***;*
5. *il dislivello tra i punti C e F* **(2 pt)***;*
6. *la lunghezza dell'arco di catenaria CF* **(2 pt)***.*



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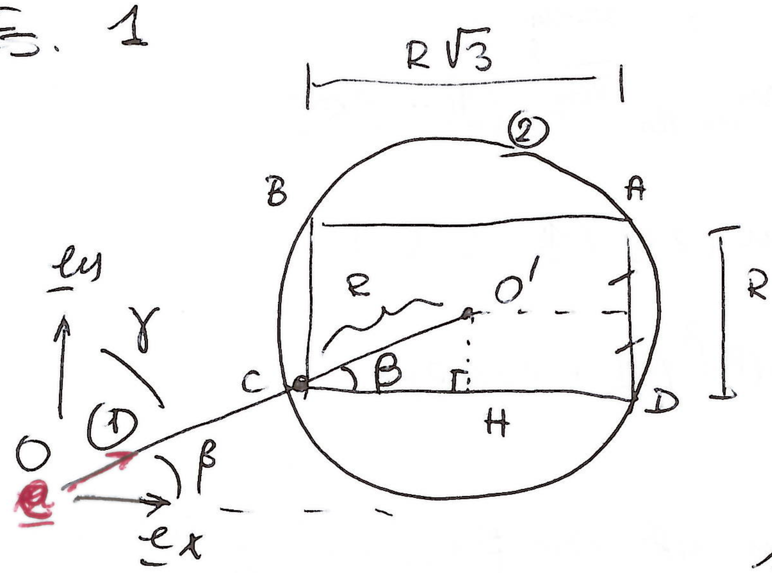
Es. 1



$$\gamma = \frac{\pi}{2} - \beta$$

$$O'H = R/2$$

$$\sin\beta = \frac{R/2}{R} = \frac{1}{2} \rightarrow \beta = \frac{\pi}{6}$$

$$\boxed{\underline{e} = \frac{\sqrt{3}}{2}\underline{e}_x + \frac{1}{2}\underline{e}_y} \quad (*)$$

$$OC = 2R - R = R$$

(1) Asta $2m$; $2R$ $\quad G_1 \equiv C$

(2) Anello $3m$; R $\quad G_2 \equiv O'$

(3) Rett. m ; $\overline{AB} = R\sqrt{3}$, $\overline{BC} = R$ $\quad G_3 \equiv O'$

} (2)+(3) $m_{23} = 4m$, $G_{23} \equiv O'$

a) Asta: $\underset{\sim}{I}_O^{(a)} = \frac{1}{3} 2m (2R)^2 (\underset{\sim}{1} - \underline{e}\otimes\underline{e})$ (*)

$$I^a_{xx} = \frac{8}{3} mR^2\, \underline{e}_x \cdot (\underset{\sim}{1} - \underline{e}\otimes\underline{e})\underline{e}_x = \frac{8}{3} mR^2 \sin^2\beta = \frac{\cancel{8}^{2}}{3} mR^2 \frac{1}{\cancel{4}} = \frac{2}{3} mR^2$$

$$I^a_{yy} = \frac{8}{3} mR^2\, \underline{e}_y \cdot (\underset{\sim}{1} - \underline{e}\otimes\underline{e})\underline{e}_y = \frac{8}{3} mR^2 \sin^2\gamma = \frac{8}{3} mR^2 \left(\frac{\sqrt{3}}{2}\right)^2 = 2mR^2$$

$$I^a_{xy} = \frac{8}{3} mR^2\, \underline{e}_x \cdot (\underset{\sim}{1} - \underline{e}\otimes\underline{e})\underline{e}_y = \frac{8}{3} mR^2 \left(-(\underline{e}\cdot\underline{e}_x)(\underline{e}\cdot\underline{e}_y)\right) =$$

$$= -\frac{\cancel{8}^{2}}{3} mR^2 \frac{\sqrt{3}}{\cancel{2}} \cdot \frac{1}{\cancel{2}} = -\frac{2}{3}\sqrt{3}\, mR^2$$

$$\left[\underset{\sim}{I}_O^{(a)}\right] = \begin{pmatrix} \boxed{\begin{matrix} 2/3 & -\frac{2}{3}\sqrt{3} \\ -\frac{2}{3}\sqrt{3} & 2 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 8/3 \end{pmatrix} mR^2$$

b) Anello

$$\underset{\sim}{I}_O^{(an)} = \underset{\sim}{I}_{G_2}^{(an)} + m_{anello}\,\overline{OG_2}^{\,2}\left(\underset{\sim}{1} - e \otimes e\right)$$

$$= \underset{\sim}{I}_{G_2}^{(an)} + 3m\,(2R)^2\left(\underset{\sim}{1} - e \times e\right)$$

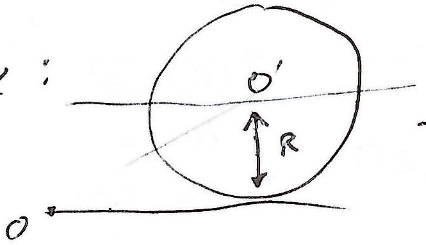
$$\underset{\sim}{I}_{G_2}^{an} = \frac{1}{2}\,3m(R)^2\left(\underset{\sim}{1} + e_z \otimes e_z\right)$$

$$I_{xx}^{(an)} = \frac{3}{2} mR^2 + 3m \cdot 4R^2 \sin^2\beta = \left(\frac{3}{2} + \cancel{12}^{\,3} \cdot \frac{1}{\cancel{4}}\right) mR^2 = \frac{9}{2} mR^2$$

$$I_{yy}^{(an)} = \frac{3}{2} mR^2 + 3m \cdot 4R^2 \cos^2\beta = \left(\frac{3}{2} + \cancel{12}^{\,3} \cdot \frac{3}{\cancel{4}}\right) mR^2 = \frac{21}{2} mR^2$$

formula H-S

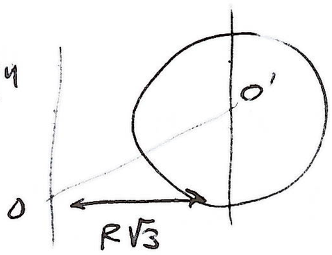
I_{xx} :



$I_{xx} = \frac{1}{2}\,3m R^2 + 3m R^2$

$I_{yy} = \frac{1}{2}\,3mR^2 + 3m(R\sqrt{3})^2$

I_{yy}



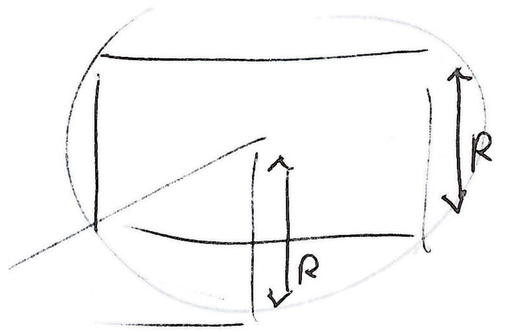
c) I_{xy}^{an} $\quad$ e_x e e_y sono direzioni principali per il C+. centrale

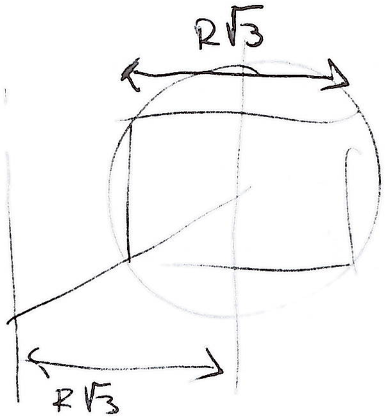
$$I_{xy}^{an} = -3m\, x_{G_2} y_{G_2} = -3m \cdot R \cdot R\sqrt{3} = -3\sqrt{3}\, mR^2$$

$$\left[\underset{\sim}{I}_O^{(an)}\right] = \begin{pmatrix} 9/2 & -3\sqrt{3} & 0 \\ -3\sqrt{3} & 21/2 & 0 \\ 0 & 0 & 15 \end{pmatrix} mR^2$$

c) Rett.

$$I_{xx}^{rett.} = \frac{1}{12} mR^2 + mR^2 = \frac{13}{12} mR^2$$





$$I_{yy} = \frac{1}{12} m (R\sqrt{3})^2 + m (R\sqrt{3})^2 =$$

$$= \frac{\cancel{3}^{1}}{\cancel{12}_{4}} mR^2 + 3mR^2 = \frac{13}{4} mR^2$$

$$I_{xy} = -m(R)(R\sqrt{3}) = -\sqrt{3}\, mR^2$$

$$[\underset{\sim}{I}_0^{tot}] = \begin{pmatrix} 13/12 & -\sqrt{3} & 0 \\ -\sqrt{3} & 13/4 & 0 \\ 0 & 0 & \frac{13}{3} \end{pmatrix} mR^2$$

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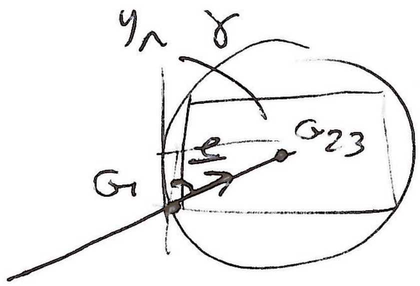
2) $\mu = \frac{m_1 \cdot m_{23}}{m_1 + m_{23}} = \frac{2m \cdot 4m}{6m} = \frac{4}{3} m$

$$\underset{\sim}{I}_G^{tot} = \underset{\sim}{I}_{G_1}^{1} + \underset{\sim}{I}_{G_{23}}^{(2,3)} + \mu d^2 (\underset{\sim}{1} - \underline{e} \otimes \underline{e})$$

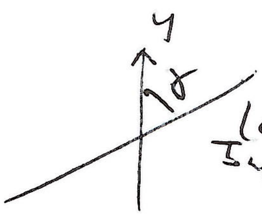


$d = R$

$\sin^2\gamma =$ [illegible]$^2\beta$

$$I_G^{yy} = I_{G_1}^{yy}(1) + I_{G_2}^{yy}(2) + \frac{4}{3} mR^2 \left(1 - (\underline{e} \cdot \underline{e}_1)^2\right) =$$

$$= \frac{\cancel{1}}{2} mR^2 + \frac{3}{2} mR^2 + \frac{1}{4} mR^2 + \frac{\cancel{4}}{\cancel{3}} mR^2 \frac{\cancel{3}}{\cancel{4}} = \boxed{\frac{13}{4} mR^2}$$



$$I_{yy}^{(G_1,\,asta)} = \frac{1}{12} 2m (2R)^2 \sin^2\gamma = \frac{\cancel{8}^{2}}{\cancel{12}_{4}} mR^2 \frac{\cancel{3}}{\cancel{4}} = \frac{1}{2} mR^2$$

Es
2.

(4)

$\underline{v}_1 = \underline{e}x - 2\underline{e}y + 3\underline{e}z$ in $P_1 \equiv (1,1,0)$

$\underline{v}_2 = \underline{e}x + \underline{e}y + \underline{e}z$ in $P_2 \equiv (0,-1,1)$

$\underline{v}_3 = -\underline{e}x + \underline{e}y + 2\underline{e}z$ in $P_3 \equiv (1,0,-1)$

a) $\underline{R} = \underline{e}x + 6\underline{e}z$

$\underline{M}_0 = (P_1 - O) \wedge \underline{v}_1 + (P_2 - O) \wedge \underline{v}_2 + (P_3 - O) \wedge \underline{v}_3 = *$

$$\begin{vmatrix} ex & ey & ez \\ 1 & 1 & 0 \\ 1 & -2 & 3 \end{vmatrix} \quad \begin{vmatrix} ex & ey & ez \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \quad \begin{vmatrix} ex & ey & ez \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$* = (3\underline{e}x - 3\underline{e}y - 3\underline{e}z) +$
$(-2\underline{e}x + \cancel{\underline{e}y} + \cancel{\underline{e}z}) +$
$(\underline{e}x - \cancel{\underline{e}y} + \underline{e}z) =$

$\underline{M}_0 = 2\underline{e}x - 3\underline{e}y - \underline{e}z$

b) $\underline{M}_Q = \underline{M}_0 + (O - Q) \wedge \underline{R}$ $\quad Q \equiv (1,1,3)$

$$\begin{vmatrix} ex & ey & ez \\ -1 & -1 & -3 \\ 1 & 0 & 6 \end{vmatrix} = -6\underline{e}x + 3\underline{e}y + \underline{e}z$$

$\underline{M}_Q = 4\underline{e}x + 0 + 0 = -4\underline{e}x$

c) $\underline{v} \perp \underline{M}_0$; $\parallel \underline{e}x$ $\quad \underline{v} = 4\underline{e}y$ applicato in P t.c.

$(P - Q) \wedge \underline{v} = \underline{M}_Q$

$(P - Q) = \dfrac{\underline{v} \wedge \underline{M}_Q}{v^2} + \lambda \underline{v} \Rightarrow \dfrac{4\underline{e}y \wedge (-4\underline{e}x)}{16} = \dfrac{16}{16} \underline{e}x \wedge \underline{e}y = \underline{e}z$

$P-O = (P-Q)+(Q-O)$

$P \equiv (0,0,1)+(1,1,3) = (1,1,4)$

$$\Sigma' = \{(Q, \underline{R}-\underline{v}), (P, \underline{v})\} =$$

$$= \{((1,1,3), \underline{e}_x - 4\underline{e}_y + 6\underline{e}_z), ((1,1,4), 4\underline{e}_y)\}$$

$R_{\Sigma'} = R_{\Sigma}$ $\qquad \underline{M}_Q(\Sigma') = \underline{M}_Q(\Sigma)$ ∎

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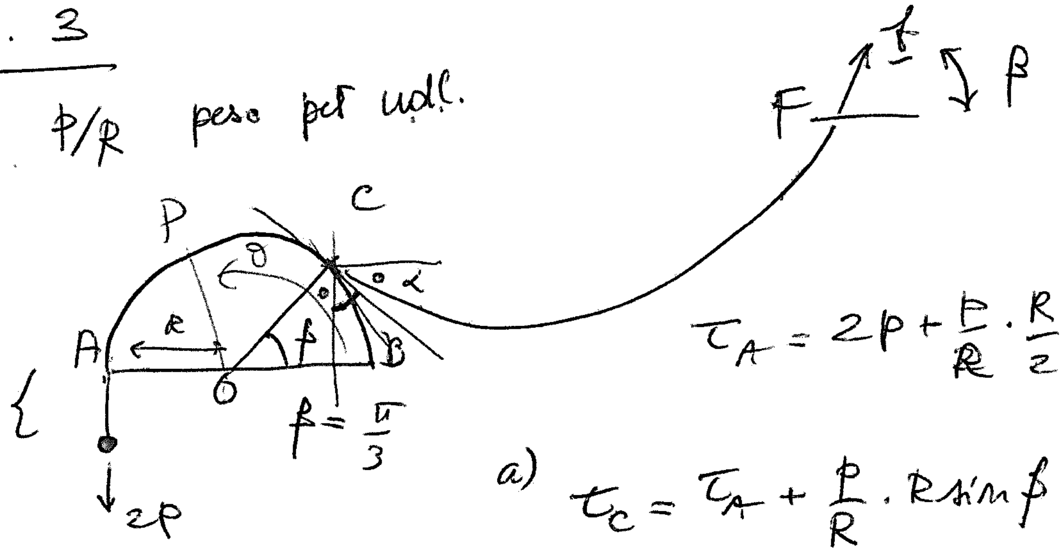
es. 3

p/R peso per udl.



$$\tau_A = 2p + \frac{p}{R}\cdot\frac{R}{2} = 2p + \frac{p}{2} = \frac{5}{2}p$$

$T = \text{cost.} + V$

a) $\tau_C = \tau_A + \frac{p}{R}\cdot R\sin f =$

$$= \frac{5}{2}p + \frac{\sqrt{3}}{2}p = \frac{5+\sqrt{3}}{2}p$$

$(\sin(\pi-\theta))$

b) $\tau(\theta) = \tau_A + \frac{p}{R}\cdot R\sin\theta = \frac{5}{2}p + p\sin\theta$

c) $\psi = \tau_x = \tau_C\cdot\cos\alpha = \tau_C \sin f = \frac{\sqrt{3}}{2}\cdot\frac{5+\sqrt{3}}{2}p = \frac{3+5\sqrt{3}}{4}p$

$\alpha = \frac{\pi}{2} - f$

$a = \frac{\psi}{p/R} = \frac{3+5\sqrt{3}}{4}R$ $\qquad y(x) = a\left[\cosh\left(\frac{x}{a}\right) - 1\right]$

$$= \frac{(3+5\sqrt{3})}{4}R\left[\mathrm{Ch}\left(\frac{4x}{(3+5\sqrt{3})R}\right) - 1\right]$$

d)

$$f = \gamma p$$

$$\tau_x = f \cdot \cos\beta = \frac{f}{2}$$

$$\frac{\gamma \cancel{p}}{\cancel{2}} = \frac{5\sqrt{3}+3}{2\,\cancel{4}}\cancel{p} \qquad \boxed{\gamma = \frac{3+5\sqrt{3}}{2}}$$

e)

$$\overset{f}{\tau_F} - \tau_C = \left(\frac{p}{R}\right)\cdot h$$

$$h = \frac{R}{\cancel{p}}\cdot\left(\frac{3+5\sqrt{3}}{2}\cancel{p} - \frac{5+\sqrt{3}}{2}\cancel{p}\right) = \frac{3+5\sqrt{3}-5-\sqrt{3}}{2}R$$

$$= \frac{4\sqrt{3}-2}{2}R = \boxed{(2\sqrt{3}-1)R}$$

f)

$$l = \int_{x_C}^{x_F}\left(1+(y'(x))^2\right)^{1/2}dx = \int_{x_C}^{x_F}\sqrt{1+Sh^2\left(\frac{x}{a}\right)}\,dx = \int_{x_C}^{x_F} Ch\left(\frac{x}{a}\right)dx =$$

$$\underline{r} = x\underline{e}_x + y(x)\underline{e}_y$$

$$\underline{r}' = 1\underline{e}_x + y'(x)\underline{e}_y$$

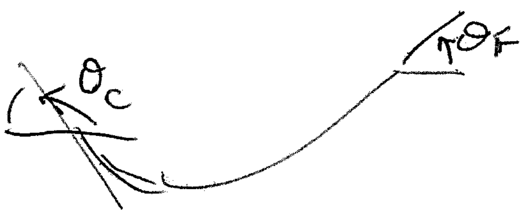
$$y'(x) = a\frac{1}{a}Sh\left(\frac{x}{a}\right) = Sh\frac{x}{a}$$

$$= tg\,\theta(x)$$

$$= a\cdot Sh\left(\frac{x}{a}\right)\Big|_{x_C}^{x_F} =$$

$$= a\left(Sh\frac{x_F}{a} - Sh\frac{x_C}{a}\right) =$$

$$= a\left(tg\,\theta_F - tg\,\theta_C\right) =$$



$$= \frac{3+5\sqrt{3}}{4}\left(\sqrt{3} - \left(-\frac{\sqrt{3}}{3}\right)\right)R$$

$$= \frac{3+5\sqrt{3}}{\cancel{4}}\left(\frac{\cancel{4}}{3}\sqrt{3}\right)R = \frac{3\sqrt{3}+15}{3}R =$$

$$= (5+\sqrt{3})R$$