Workshop:
Recent advances in discontinuous Galerkin methods
Programme and Abstracts

University of Reading

Monday 13th June 2016

9:50	Opening
10:00	**Alexandre Ern**
10:30	**Erik Burman**
11:00	Coffee break
11:30	**Thomas Wihler**
12:00	**Sandra May**
12:30	**Frederick Qiu**
13:00	Lunch
14:00	**Manolis Georgoulis**
14:30	**Claire Scheid**
15:00	Coffee break
15:30	**Paola Antonietti**
16:00	**Jennifer Ryan**
16:30	**Charalambos Makridakis**
17:00	Poster session and reception (Maths common room)

All talks will take place in room 113 of the Department of Mathematics and Statistics.

Abstracts of the talks

Fast solution techniques for high order Discontinuous Galerkin methods

PAOLA ANTONIETTI, MOX, Politecnico di Milano

We present two-level and multigrid algorithms for the efficient solution of the linear system of equations arising from high-order discontinuous Galerkin discretizations of second-order elliptic problems. Starting from a classical framework in geometric multigrid analysis, we define a smoothing and an approximation property, which are used to prove uniform convergence of the resulting multigrid schemes with respect to the discretization parameters and the number of levels, provided the number of smoothing steps is chosen sufficiently large. A discussion on the effects of employing inherited or noninherited sublevel solvers is also presented as well the extension of the proposed techniques to agglomeration-based multigrid solvers. Numerical experiments confirm the theoretical results.

Nonconforming finite element methods for ill-posed problems

ERIK BURMAN, University College London

The discontinuous Galerkin method was originally introduced for its excellent stability properties for hyperbolic problems that did not fit the standard theoretical framework for elliptic problems using Lax-Milgram's and Cea's Lemma. On the other hand it is well known that the large space leads to stability problems when approximating elliptic equations, typically requiring additional stabilising terms. In this talk we are interested in ill-posed elliptic problems and we will discuss how the use of nonconforming spaces influences the numerical stability and the need of Tikhonov regularisation. As particular examples we will discuss the elliptic Cauchy problem and a data assimilation problem subject to Stokes equations. For these cases we report error estimates that can be considered optimal with respect to approximation and the problem specific conditional stability.

Hybrid high-order methods

Alexandre Ern, CERMICS Paris

Hybrid High-Order (HHO) methods have been introduced recently in [D. Di Pietro, A. Ern and S. Lemaire, Comp. Methods Appl. Math., 14(4):461–472 (2014)], [D. Di Pietro and A. Ern, Comp. Meth. Appl. Mech. Eng., 283:1–21 (2015)]. These methods are based on discrete unknowns that are discontinuous polynomials on the mesh skeleton. Such methods present several attractive features. The construction is dimension-independent, it can be deployed for arbitrary polynomial orders, and general grids, including non-matching interfaces or polyhedral cell shapes, can be used. Positioning unknowns at mesh faces allows one to work with the primal formulation of the problem and, at the same time, it is also a natural way to express at the discrete level fundamental continuum properties such as local mass or force balance. The cornerstone of the construction are fully local, problem-dependent, reconstruction operators. This approach can offer reduced computational costs by organizing simulations into (fully parallelizable) local solves and a global transmission problem. In addition, numerical analysis, confirmed by numerical examples, shows the robustness of the approach in various practically-relevant regimes where some model parameters take large values. HHO methods can be linked to the Hybridizable Discontinuous Galerkin (HDG) setting. Two distinctive features of HHO methods are a smaller space for the cell-based unknowns and a novel, parameter-free, high-order, face-based stabilization on general meshes.

Space-time DG methods for evolution PDEs

Manolis Georgoulis, University of Leicester and NTU Athens

I will review some recent work on space-time discontinuous Galerkin methods (dG) for parabolic and for second-order hyperbolic PDE problems. Space-time dG methods offer a number of advantages in these context, most important of which is the ability to choose local bases on a combined space-time fashion, which can often lead to complexity reduction as the order of the method grows.

On the L^2 stability of the discontinuous Galerkin elliptic projection

Charalambos Makridakis, University of Sussex

We discuss the stability properties of the discontinuous Galerkin elliptic projection in mesh dependent L^2 norms in unstructured meshes. As a result we show new a priori error estimates in L^2.

Entropy-stable spacetime discontinuous Galerkin methods for the compressible Navier-Stokes equations

Sandra May, ETH Zurich

We present methods for solving systems of hyperbolic conservation laws with physical diffusion terms. In particular we are interested in solving the compressible Navier-Stokes equations. Our methods are extensions of a spacetime discontinuous Galerkin method for solving hyperbolic conservation laws.

In our extension to problems with physical diffusion terms we mostly follow the original scheme for the treatment of the non-linear terms: we use entropy variables as degrees of freedom and use entropy stable numerical fluxes. For incorporating the diffusion terms, we use the interior penalty method resulting in the extensions ST-NIPG and ST-SIPG. To guarantee stability of the methods both for fine and coarse grids, we also adjust the shock capturing terms of the original method appropriately to account for the presence of the diffusion term.

In this talk, we first give a short summary of the original method for conservation laws. In the main part of the talk, we will present our new extensions and discuss their entropy stability properties. We conclude with numerical results for the two-dimensional compressible Navier-Stokes equations.

A superconvergent HDG method for the Incompressible Navier–Stokes Equations on general polyhedral meshes

FREDERICK QIU, City University of Hong Kong

We present a superconvergent hybridizable discontinuous Galerkin (HDG) method for the steady-state incompressible Navier–Stokes equations on general polyhedral meshes. For arbitrary conforming polyhedral mesh, we use polynomials of degree $k+1$, k, k to approximate the velocity, velocity gradient and pressure, respectively. In contrast, we only use polynomials of degree k to approximate the numerical trace of the velocity on the interfaces. Since the numerical trace of the velocity field is the only globally coupled unknown, this scheme allows a very efficient implementation of the method. For the stationary case, and under the usual smallness condition for the source term, we prove that the method is well defined and that the global L2-norm of the error in each of the above-mentioned variables and the discrete H1-norm of the error in the velocity converge with the order of $k+1$ for $k \leq 0$. We also show that for $k \geq 1$, the global L2-norm of the error in velocity converges with the order of $k+2$. From the point of view of degrees of freedom of the globally coupled unknown: numerical trace, this method achieves optimal convergence for all the above-mentioned variables in L2-norm for $k \geq 0$, superconvergence for the velocity in the discrete H1-norm without postprocessing for $k \geq 0$, and superconvergence for the velocity in L2-norm without postprocessing for $k \geq 1$.

Applications of multi resolution analysis in discontinuous Galerkin methods

JENNIFER RYAN, University of East Anglia

In this talk, we present a generalized discussion of discontinuous Galerkin methods concentrating on a basic concept: exploiting the existing approximation properties. The discontinuous Galerkin method uses a piecewise polynomial approximation to the variational form of a PDE. It uses polynomials up to degree k for a $k+1$ order accurate scheme. Using this formulation, we concentrate on nonlinear hyperbolic equations and specifically discuss how to obtain better discontinuity detection during time integration by rewriting the approximation using a multi-wavelet decomposition. We demonstrate that this multi-wavelet expansion allows for more accurate detection of discontinuity locations. One advantage of using the multi-wavelet expansion is that it allows us to specifically relate the jumps in the DG solution and its derivatives to the multi-wavelet coefficients. This is joint work with Thea Vuik, TU Delft.

A discontinuous Galerkin framework for the numerical modelling in nanoplasmonics

CLAIRE SCHEID, INRIA—Université de Nice Sophia Antipolis

Nanoplasmonics is the domain of physics that takes advantage of the effects of the interaction of light with nanometer scaled metallic structures. A lot of interesting applications emerge nowadays, ranging from nanolasers to subwavelength imaging. Besides experiments, numerical modelling is essential and there is many challenges to tackle. We thus look for reliable discretization methods that could be flexible enough to undertake the geometrical, physical and scales complexities. In this context, we introduce a numerical framework based on a Time Domain Discontinuous Galerkin approach that is adapted to the models encountered in nanoplasmonics. We propose a complete study from the numerical analysis to realistic numerical tests that assess the validity of this numerical method.

An hp-Adaptive Newton-Discontinuous-Galerkin (NDG) Finite Element Approach for Semilinear Elliptic Boundary Value Problems

THOMAS WIHLER, University of Bern

We develop an hp-adaptive procedure for the numerical solution of general, semilinear elliptic boundary value problems, with possible singular perturbation. Our approach combines both adaptive Newton schemes and an hp-version adaptive discontinuous Galerkin finite element discretisation, which, in turn, is based on a robust hp-version a posteriori residual analysis.

Abstracts of the posters

A hybrid discontinuous Galerkin inspired preconditioner for the Stokes equation with non-standard boundary conditions

MICHAL BOSY, University of Strathclyde

We consider the Stokes problem with non-standard boundary conditions. Specifically, we consider a Stokes problem with an imposed tangential velocity and a normal flux on the boundary. The discretisation is carried out using a hybrid discontinuous Galerkin method. Moreover, we introduce a new domain decomposition preconditioner that is easy to parallelise. Finally, we present convergence validation of the problem and we later compare the proposed preconditioner with standard Restricted Additive Schwarz preconditioners.

Boundary element method for high frequency scattering by multiple objects

ANDREW GIBBS, University of Reading

We propose a boundary element method developed for problems of scattering by multiple objects, at least one of which is a convex polygon. Focusing on the specific example of two scatterers of significantly different size, we use oscillatory basis functions on the large scatterer and piecewise Legendre polynomials on the small scatterer. In the case of a single scatterer, the hybrid basis consisting of oscillatory basis functions is based on an exact representation on the boundary. This has been shown to have only logarithmic dependence on the frequency of the incident wave. An analogous representation is derived for the case of multiple scatterers, which is then utilised in this method.

Hybridised methods for ocean and atmosphere discretisations

THOMAS GIBSON, Imperial College London

Efficient solution of geophysical flow problems depends on a Schur complement transformation of the mixed velocity-pressure (or velocity-pressure- buoyancy) system into an elliptic system for pressure. Implementing this transformation for H(div) velocity spaces requires “hybridisation” techniques, in which the continuity of the velocity normals is imposed via an explicit flux variable. As part of the Firedrake Project, we introduce the necessary abstractions and operations required for assembling the “hybridised” system. By introducing symbolic operators for the hybridisation operations, and generating code from them, we aim to successfully extend this system for use in automated simulation to address this important class of problem.

Joint work with David A. Ham and Colin J. Cotter.

Conservative numerical schemes for wave equations

JAMES JACKAMAN, University of Reading

Wave equations are useful tools, the equations we consider, Korteweg-de Vries (KdV) and Camassa-Holm (CH), can be derived from the Navier-Stokes equations. We numerically approximate these equations using dG methods, and find that our numerical solution preserves discrete versions of the physical properties under particular flux choices.

Multiresolution shallow water modelling

GEORGES KESSERWANI, University of Sheffield

Numerical modelling of wide ranges of different physical scales, which are involved in Shallow Water (SW) problems, has been a key challenge in computational hydraulics. Adaptive meshing techniques have been commonly coupled with numerical methods in an attempt to address this challenge. The combination of MultiWavelets (MW) with the RungeKutta Discontinuous Galerkin (RKDG) method offers a new philosophy to readily achieve mesh adaptivity driven by the local variability of the numerical solution, and without requiring more than one threshold value set by the user. However, the practical merits and implications of the MWRKDG, in terms of how far it contributes to address the key challenge above, are yet to be explored. This work systematically explores this, through the verification and validation of the MWRKDG for selected steady and transient benchmark tests, which involves the features of real SW problems. Our findings reveal a practical promise of the SW-MWRKDG solver, in terms of efficient and accurate mesh-adaptivity, but also suggest further improvement in the SW-RKDG reference scheme to better intertwine with, and harness the prowess of, the MW-based adaptivity.