Workshop:
Recent advances in discontinuous Galerkin methods
Programme and Abstracts

University of Reading

11th–12th September 2014

Thursday	September 11th
12:00	Lunch (Maths common room)
12:50	Opening (Simon Chandler-Wilde)
13:00	**Paul Houston**
13:30	**Edward Hall**
14:00	**Emmanuil Georgoulis**
14:30	**Matthias Maischak**
15:00	Coffee break
15:30	**Charalambos Makridakis**
16:00	**Andreas Dedner**
16:30	**Iain Smears**
17:00	**Omar Lakkis**
17:30	Poster session and reception (Maths common room)
19:00	Social dinner at Queen's Head

Friday	September 12th
9:00	**Herbert Egger**
9:30	**Blanca Ayuso de Dios**
10:00	**Andrea Cangiani**
10:30	**Irene Kyza**
11:00	Coffee break
11:30	**Gabriel Barrenechea**
12:00	**Stefano Giani**
12:30	**Foteini Karakatsani**
13:00	**Tristan Pryer**
13:30	Lunch

All talks will take place in room 113 of the Department of Mathematics and Statistics.

Abstracts of the talks

Auxiliary Space Preconditioners for discontinuous Galerkin discretizations of H(curl)-elliptic problems.

BLANCA AYUSO DE DIOS, KAUST, Saudi Arabia

We introduce a family of preconditioners for discontinuous Galerkin discretizations of H(curl)-elliptic problems with (possibly) jumping coefficients. The design and analysis of the proposed solvers is based on the auxiliary space method. The preconditioners are shown to be robust with respect to the coefficients. We validate the theory with extensive numerical experiments. Joint work with Ralf Hiptmair and Cecilia Pagliantini from ETH (Zürich).

Recent results on positivity-preserving schemes for the convection-diffusion equation

GABRIEL BARRENECHEA, University of Strathclyde

The numerical approximation of the convection–diffusion equation is known to be challenging, especially in the convection-dominated regime. A major drawback is the lack of positivity preserving schemes, i.e., schemes that satisfy the discrete maximum principle. Since linear schemes satisfying this property are known to lead to extremely diffusive results, an alternative approach has been to introduce nonlinear schemes, mostly inspired by the idea of shock capturing, to solve this equation. One scheme that has received some attention over the last few years is the algebraic flux correction scheme. The origins of this method can be traced back to the late eighties, but they have been reframed recently in the works by D. Kuzmin. In this talk I will review some recent analytical results for this scheme, showing its advantages and limitations, and present a modification of it that makes it possible to prove that the discrete problem has a solution. Some preliminary numerical results are also presented. This work has been carried out in collaboration with Volker John (WIAS, Berlin) and Petr Knobloch (Charles University, Prague).

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hp-Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes

ANDREA CANGIANI, University of Leicester

In this talk we present the hp-version interior penalty discontinuous Galerkin method for the discretization of linear partial differential equations on general computational meshes consisting of polygonal/polyhedral elements. Two model problems are considered: the Poisson problem and the first order transport problem. The method employs elemental polynomial bases of total degree p defined on the physical space, without the need to map from a given reference or canonical frame. In the case of second order elliptic problems, a new specific choice of the interior penalty parameter which allows for face-degeneration is presented. New refined inverse inequalities ensure that optimal a priori bounds may be established, for general meshes including polygonal elements with degenerating edges in two dimensions and polyhedral elements with degenerating faces and/or edges in three dimensions. Numerical experiments highlighting the performance of the proposed method are presented. This is joint work with Manolis Georgoulis and Zhaonan Dong (Leicester) and Paul Houston (Nottingham).

Discontinuous Galerkin Methods for Surface PDEs

ANDREAS DEDNER, University of Warwick

We will present a-priori error estimates for DG schemes for advection–diffusion problems on surfaces. The surface finite-element method with continuous ansatz functions was analysed some time ago and we will discuss how to extend these results to a wide range of DG methods where the non-smooth approximation of the surface introduces some additional challenges. Both a-priori and a-posteriori analysis is presented for some model problems together with numerical experiments.

A posteriori estimates for hybrid discontinuous Galerkin methods

HERBERT EGGER, TU Darmstadt, Germany

We discuss a class of a-posteriori error estimators for hybrid discontinuous Galerkin methods. As two particular instances, we obtain an estimator of residual-type, and a second estimator based on equilibration. Both methods yield certified upper bounds for the error without generic constants and only a small sub-optimality with respect to the polynomial degree. We also discuss the extension to the Stokes system.

Space-time discontinuous Galerkin methods for the wave equation

EMMANUIL GEORGOULIS, University of Leicester

I will start by reviewing some (classical) space-time finite element methods for wave problems, along with some challenges posed by attempts to provide error analysis for them. I will continue with a new formulation of space-time discontinuous Galerkin method, which will be shown to be stable and dissipative. A key motivation for such a space-time discontinuous method is its possible use in conjunction with basis functions which are themselves solutions to the wave equation—a 'Trefftz-type' approach. Some comments on error analysis for this class of methods will be given. This is joint with with Lehel Banjai (Heriott-Watt).

High-Order/hp-Adaptive Multilevel Discontinuous Galerkin Methods

STEFANO GIANI, Durham University

We present a discontinuous Galerkin (DG) multilevel method with hp-adaptivity. The main advantage of this multilevel method is that the number of dimensions of the finite element space is independent on the presence of complicated or tiny features in the domain. In other words, even on a very complicated domain, an approximation of the solution can be computed with only a fistful of degrees of freedom. This is possible because two meshes are used: a fine mesh is used to describe the geometry of the domain with all its features, but the problem is actually solved on a coarse mesh that is, in general, too coarse to describe all the geometrical features of the domain. Unlikely other multilevel methods, this method does not perturb the problem, in the sense that the problem solved on the coarse mesh is always a discretization of the continuous problem, no matter how coarse the mesh is. The method itself is a hp-adaptive DG extension of composite finite elements (CFEs), introduced by S. Sauter a few years ago. Standard CFE methods are based on standard continuous Galerkin elements, which means that there are restrictions on the kind of boundary conditions that can be used. These limitations disappear by extending the method to DG elements.

The hp-adaptivity algorithm that we present for this multilevel method is completely automatic and capable of exploiting both local polynomial-degree-variation (p-refinement) and local mesh subdivision (h-refinement), thereby offering greater flexibility and efficiency than numerical techniques which only incorporate h-refinement or p-refinement alone.

This research has been funded by the EPSRC.

Robust Reconstruction Based A Posteriori Error Bounds For DG Method

EDWARD HALL, University of Leicester

Higher order reconstructions have previously been used to give a posteriori error bounds in the context of DG time stepping, see for example [1]. In this talk, we present the general principles behind the reconstruction technique and show how they can be used for steady 1D convection–diffusion–reaction problems. Crucially, we show that the a posteriori error bounds are robust in the hyperbolic limit and present some numerical examples to highlight this. We then discuss the use of reconstructions in the 2D dimensional setting and, if time allows, their extension to time dependent problem using the RKDG method.

Joint work with E.H. Georgoulis and C. Makridakis.

[1] C. Makridakis and R. H. Nochetto, *A posteriori error analysis for higher order dissipative methods for evolution problems*. Numer. Math., 104 (2006), pp. 489–514

[2] E.H. Georgoulis, E. Hall and C. Makridakis, *Error control for Discontinuous Galerkin methods for first order hyperbolic problems*. In Barrett Lectures on Discontinuous Galerkin Methods, Springer, 2014

[3] E.H. Georgoulis, E. Hall and C. Makridakis, *An a posteriori error bound for Discontinuous Galerkin approximations of convection-diffusion problems*. Submitted for publication.

Robust hp–Version Discontinuous Galerkin Finite Element Methods for Nonlinear PDEs

PAUL HOUSTON, University of Nottingham

In this talk we present an overview of some recent developments concerning the a posteriori error analysis and adaptive mesh design of h– and hp–version discontinuous Galerkin finite element methods for the numerical approximation of second-order quasilinear elliptic boundary value problems. In particular, we consider the derivation of computable bounds on the error measured in terms of an appropriate (mesh-dependent) energy norm in the case when a two-grid approximation is employed. In this setting, the underlying nonlinear problem is first computed on a coarse finite element space $V_{H,P}$. The resulting 'coarse' numerical solution is then exploited to provide the necessary data needed to linearise the proposed discretization on the finer space $V_{h,p}$; thereby, only a linear system of equations is solved on the richer space $V_{h,p}$. Here, an adaptive hp–refinement algorithm is proposed which automatically selects the local mesh size and local polynomial degrees on both the coarse and fine spaces $V_{H,P}$ and $V_{h,p}$, respectively. Numerical experiments confirming the reliability and efficiency of the proposed mesh refinement algorithm are presented.

A posteriori error analysis of fully discrete schemes for time-dependent Stokes equations

FOTEINI KARAKATSANI, University of Strathclyde

We derive residual-based a posteriori error estimates of optimal order for fully discrete approximations of the time-dependent Stokes problem. The time discretization uses the backward Euler method and the spatial discretization uses finite element spaces that are allowed to change in time. The a posteriori error estimates are derived by applying the reconstruction technique.

[1] E. Bänsch, F. Karakatsani, Ch. Makridakis, *On the a posteriori error control of a time dependent Stokes equations*. Preprint (2013).

[2] F. Karakatsani, Ch. Makridakis, *A posteriori estimates for approximations of time dependent Stokes equations*. IMA J. Numer. Anal. 27 (2007) 741–764.

[3] O. Lakkis, Ch. Makridakis, *Elliptic reconstruction and a posteriori error estimates for fully discrete linear parabolic problems*. Math. Comp. 75 (2006) 1627-1658.

Adaptivity and blowup detection for an evolution semilinear convection–diffusion model problem

IRENE KYZA, University of Dundee

We discuss recent results on the a posteriori error control and adaptivity for an evolution semilinear convection–diffusion model problem with possible blowup in finite time. This belongs to the broad class of partial differential equations describing e.g., tumor growth, chemotaxis and cell modelling. The analysis is performed using the reconstruction technique. In particular, we derive a posteriori error estimates that are conditional (estimates which are valid under conditions of a posteriori type) for an interior penalty discontinuous Galerkin (dG) implicit-explicit (IMEX) method using a continuation argument. Compared to a previous work, the obtained conditions are more localised and allow the efficient error control near the blowup time. Utilising the conditional a posteriori estimator we are able to propose an adaptive algorithm that appears to perform satisfactorily. In particular, it leads to good approximation of the blowup time and of the exact solution close to the blowup. Numerical experiments illustrate and complement our theoretical results. This is joint work with A. Cangiani, S. Metcalfe, and E.H. Georgoulis from the University of Leicester.

A posteriori analysis and adaptivity in discontinuous Galerkin time-stepping methods

OMAR LAKKIS, University of Sussex

A posteriori error analysis for time-DG discretization of parabolic problems requires special reconstructions both in space and time. Building on previous work by Makridakis & Nochetto and Schötzau & Whiler, we derive error estimates for fully discrete schemes for the maximum-mean square norm and mean-square-energy norm.

Joint work with E.H. Georgoulis, D. Schötzau and T. Whiler.

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hp-Discontinuous Galerkin method for an Elliptic–Hyperbolic coupling problem

MATTHIAS MAISCHAK, Brunel University

We use the hp-discontinuous Galerkin method to solve a nonlinear elliptic–hyperbolic coupling problem modeling Corona discharge in 2d and 3d. We give a-priori error estimates based on regularity results. We investigate several linearization techniques and their consequences for the discretization and effects on the solution process. We also give a-posteriori error estimates. The hp-discontinuous Galerkin method will be compared with other approaches, e.g. Petrov-Galerkin, and numerical examples (model problems and realistic experimental settings) will be presented which underline the theoretical results.

Error control for nonlinear hyperbolic systems

CHARALAMBOS MAKRIDAKIS, University of Sussex

We present an overview of estimators along with new results for DG methods approximating hyperbolic conservation laws. The results in the scalar case are based on Kruzkov's estimates, while in the case of systems are based on relative entropy techniques.

A posteriori analysis of some inconsistent, nonconforming Galerkin methods approximating elliptic problems

TRISTAN PRYER, University of Reading

We discuss a methodology for the a posteriori analysis of inconsistent finite element schemes for elliptic problems. We develop this for a model second order linear elliptic problem and move on to extend to nonlinear second and fourth order problems of p-Laplace and p-biharmonic type. We pay particular attention in examining the nature of the a posteriori inconsistency error and draw relations to the a priori case.

***hp*-version DGFEM for elliptic and parabolic Hamilton–Jacobi–Bellman equations with Cordes coefficients**

IAIN SMEARS, University of Oxford

We will present theoretical and computational aspects of the hp-version DGFEM for fully nonlinear second-order elliptic and parabolic Hamilton–Jacobi–Bellman equations with Cordes coefficients, which is joint work with Endre Süli. The discretisation of the PDE is motivated by its continuous analysis that is based on the Cordes condition and that establishes well-posedness in the class of strong solutions. We will see that the methods are consistent and stable, with error bounds that are optimal in the mesh size, and suboptimal in the polynomial degrees, as standard for hp-version DGFEM. Numerical experiments on problems with strongly anisotropic diffusion coefficients study the performance of the schemes and show their efficient discrete solution by a combination of semismooth Newton methods and nonoverlapping domain decomposition preconditioners.

Abstracts of the posters

Estimation of arbitrary order central statistical moments by the Multilevel Monte Carlo Method

CLAUDIO BIERIG, University of Reading

We extend the general framework of the Multilevel Monte Carlo method to multilevel estimation of arbitrary order moments. In particular, we prove that under certain assumptions, the total cost of the MLMC moment estimator is asymptotically the same as the cost of the multilevel sample mean estimator and thereby is asymptotically the same as the cost of a single deterministic forward solve. The general convergence theory is applied to a class of obstacle problems with rough random obstacle profiles. Numerical experiments confirm theoretical findings.

Hybrid BEM for scattering by multiple convex polygons

ANDREW GIBBS, University of Reading

We propose a hybrid numerical-asymptotic boundary element method for problems of high frequency scattering by multiple convex polygons. Standard numerical schemes for scattering problems have a computational cost that grows at least in direct proportion to the frequency of the incident wave. For many problems of scattering by single obstacles, it has been shown that a careful choice of approximation space, utilising knowledge of high frequency asymptotics, can lead to numerical schemes whose computational cost is independent of frequency. Here, we extend these ideas to multiple scattering configurations, focusing in particular on the case where one obstacle is much larger than the others.

A high frequency BEM for scattering by penetrable obstacles

SAMUEL GROTH, University of Reading

We present a hybrid numerical-asymptotic boundary element method for the problem of high frequency scattering by penetrable polygonal obstacles. Standard boundary element schemes based on piecewise polynomial approximation spaces suffer from the requirement that the number of degrees of freedom must increase linearly (in 2D) with respect to the frequency to maintain a prescribed accuracy. High frequency asymptotics, on the other hand, are non-convergent and may be inaccurate at low to medium frequencies. The so-called hybrid numerical-asymptotic method combines the best features of both approaches by enriching the approximation space with oscillatory functions carefully chosen using knowledge of the high frequency behaviour of the solution. We describe the salient features of this method and present results demonstrating that, for absorbing scatterers, we can achieve a prescribed accuracy as frequency increases without increasing the number of degrees of freedom.

The Space–Time Discontinuous Galerkin Trefftz Method

FRITZ KRETZSCHMAR, TU Darmstadt, Germany

The Discontinuous Galerkin Trefftz Finite Element Method is a novel type of DG method that employs space-time Trefftz basis functions which exactly satisfy the underlying partial differential equations. As a result spectral convergence in the whole space-time domain of interest is obtained. In this work we introduce the essence of the method and present two different choices of Trefftz bases.

Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations.

XIONG MENG, University of East Anglia

We analyze discontinuous Galerkin methods using upwind-biased numerical fluxes for time-dependent linear conservation laws. In one dimension, optimal a priori error estimates of order $k+1$ are obtained when piecewise polynomials of degree at most k ($k \geq 0$) are used. Our analysis is valid for arbitrary nonuniform regular meshes and for both periodic boundary conditions and for initial-boundary value problems. We extend the analysis to the multidimensional case on Cartesian meshes when piecewise tensor product polynomials are used. Numerical experiments are shown to demonstrate the theoretical results.

Stabilized Galerkin methods for the advection of differential forms with discontinuous velocity fields

CECILIA PAGLIANTINI, ETH Zürich, Switzerland

Aiming at solving the problem of resistive magnetohydrodynamics (MHD), a numerical treatment of the equations governing the evolution of electromagnetic fields based on a stabilized Galerkin approach in the spirit of discontinuous Galerkin methods is proposed. The MHD equations are a widely used macroscopic model for the behavior of conducting fluids, like plasma, interacting with electromagnetic fields. Due to the strongly non-linear hyperbolic character of the system, the inevitable shock formation introduces discontinuities into the velocity field. For this reason, the quasi magneto-static Maxwell's equations in the limit of small diffusivity are of specific interest in the case of given discontinuous transport velocities.

In this context, the more general transport problem for a differential k-form is numerically tackled with implicit time integrators and spatial discretizations relying on both conforming and non-conforming discrete differential forms. The numerical analysis of the method is quite challenging on account of lacking well-posedness results for the continuous problem when $k \geq 1$. The scheme we propose is an extension of the Eulerian conforming stabilized Galerkin method introduced in [1] for stationary magnetic advection with Lipschitz continuous velocity fields.

Numerical examples in 2D on both structured and unstructured meshes for the steady-state and transient advection problem with discontinuities in the velocity are provided.

This work is part of a PhD project under the supervision of Prof. R. Hiptmair and Prof. S. Mishra. Support by the Swiss NSF Grant No. 146355.

[1] H. Heumann and R. Hiptmair, *Stabilized Galerkin methods for magnetic advection*. ESAIM: Math. Model. Numer. Anal., 47:1713–1732, 2013.

Sparse space-time Galerkin BEM for the nonstationary heat equation

ANNE REINARZ, University of Reading

The numerical solution of parabolic time evolution problems such as the heat equation is required in numerous applictions. Solving this problem using the boundary element method is an attractive alternative to traditional methods, such as combining Finite Elements or Finite Differences with a variety of time-stepping schemes. The boundary element approach to this problem is well understood. Results on mapping properties of the operators, regularity and coercivity of the formulation are known [Costabel, 1990]. This poster presents the idea of combining the boundary element method with anisotropic sparse tensor product bases that can yield better convergence rates [Chernov, Schwab, 2011]. This is expected to reduce the total work to $O(h_x^{-(d-1)})$, where d is the spatial dimension. Wavelet compression results can be used to ensure that the resulting linear system can be solved in linear complexity. The theoretical results are supported by numerical experiments.

Numerical integration of Landau–Lifshitz–Gilbert equation

MICHELE RUGGERI, TU Wien, Austria

The understanding of the magnetization dynamics, especially on a microscale, plays an important role in the design of many technological applications, e.g., magnetic sensors, recording heads, and magneto-resistive storage devices. In micromagnetism, it is well-accepted that the partial differential equation governing the dynamics of the magnetization is the Landau-Lifshitz-Gilbert equation (LLG). The reliable numerical integration of LLG faces several challenges due to the strong nonlinearity, possibly complicated and nonlocal field contributions, as well as an inherent non-convex side constraint which enforces length preservation. In general, when dealing with partial differential equations characterized by pointwise constraints, several numerical methods employ a nodal projection step. More precisely, at each time step the constraint is posed at the nodes of the triangulation via a projection of the computed solution onto the target manifold under consideration. To prove stability of such methods, some restrictive conditions on the step size or the underlying triangulation need to be imposed to allow the use of monotonicity arguments. We consider the tangent plane scheme from [2]. We show that the nodal projection step is actually not necessary for the finite element approximation to converge towards a weak solution of LLG. This leads to a violation of the constraint at the nodes of the triangulation, which is however controlled by the time step size, independently of the number of iterations. In particular, our analysis can therefore avoid a technical angle condition on the triangulation. Both versions of the integrator are computationally attractive and suitable for the development of decoupled algorithms for the numerical solution of coupled systems where LLG is coupled with another partial differential equation. As an example, we consider a multiscale model, where LLG is coupled with the magnetostatic Maxwell equations [3], and a spin-polarized transport model, where the coupling of LLG with a diffusion equation for the spin accumulation field is considered [1].

This research has been supported by the Austrian Science Fund (FWF) under grant W1245, and through the innovative projects initiative of Vienna University of Technology.

[1] C. Abert, G. Hrkac, M. Page, D. Praetorius, M. Ruggeri, and D. Suess. *Spin-polarized transport in ferromagnetic multilayers: An unconditionally convergent FEM integrator*. Comput. Math. Appl., http://dx.doi.org/10.1016/j.camwa.2014.07.010, 2014.

[2] F. Alouges. *A new finite element scheme for Landau-Lifchitz equations*. Discrete Contin. Dyn. Syst. Ser. S, 1(2):187196, 2008.

[3] F. Bruckner, M. Feischl, T. Führer, P. Goldenits, M. Page, D. Praetorius, M. Ruggeri, and D. Suess. *Multiscale modeling in micromagnetics: Existence of solutions and numerical integration*. Math. Models Methods Appl. Sci., http://dx.doi.org/10.1142/S0218202514500328, 2014.