On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.

You will be given five minutes at the end of the examination to complete the front of any answer books used.

May/June 2014 **MA3VC 2013/4 A 001**

Any non-programmable calculator permitted

UNIVERSITY OF READING

VECTOR CALCULUS (MA3VC)

Two hours

Answer **ALL** questions. Total marks: 50.

You may use the following identities in the solution of the exercises:

$$\begin{aligned}
\vec{\nabla}(fg) &= f\vec{\nabla}g + g\vec{\nabla}f, \\
\vec{\nabla}(\vec{\mathbf{F}}\cdot\vec{\mathbf{G}}) &= (\vec{\mathbf{F}}\cdot\vec{\nabla})\vec{\mathbf{G}} + (\vec{\mathbf{G}}\cdot\vec{\nabla})\vec{\mathbf{F}} + \vec{\mathbf{G}}\times(\vec{\nabla}\times\vec{\mathbf{F}}) + \vec{\mathbf{F}}\times(\vec{\nabla}\times\vec{\mathbf{G}}), \\
\vec{\nabla}\cdot(f\vec{\mathbf{G}}) &= (\vec{\nabla}f)\cdot\vec{\mathbf{G}} + f\vec{\nabla}\cdot\vec{\mathbf{G}}, \\
\vec{\nabla}\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) &= (\vec{\nabla}\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec{\nabla}\times\vec{\mathbf{G}}), \\
\vec{\nabla}\times(f\vec{\mathbf{G}}) &= (\vec{\nabla}f)\times\vec{\mathbf{G}} + f\vec{\nabla}\times\vec{\mathbf{G}}, \\
\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) &= (\vec{\nabla}\cdot\vec{\mathbf{G}})\vec{\mathbf{F}} - (\vec{\nabla}\cdot\vec{\mathbf{F}})\vec{\mathbf{G}} + (\vec{\mathbf{G}}\cdot\vec{\nabla})\vec{\mathbf{F}} - (\vec{\mathbf{F}}\cdot\vec{\nabla})\vec{\mathbf{G}}, \\
\Delta(fg) &= (\Delta f)g + 2\vec{\nabla}f\cdot\vec{\nabla}g + f(\Delta g).
\end{aligned}$$

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1. (a) Let f and g be two smooth scalar fields. Prove the following identity:

$$\vec{\nabla}\times(f\vec{\nabla}g) + \vec{\nabla}\times(g\vec{\nabla}f) = \vec{\mathbf{0}}.$$

You can use the vector differential identities proved in class; otherwise it is sufficient to prove the equality for the x-component only.

[7 marks]

(b) Demonstrate the above identity for the following choice of the fields:

$$f(\vec{\mathbf{r}}) = x^2y^3, \qquad g(\vec{\mathbf{r}}) = \sin 2z - \cos x.$$

[7 marks]

(c) From the identity above, it follows that $\vec{\nabla}\times(f\vec{\nabla}f) = \vec{\mathbf{0}}$ for all smooth scalar fields f. Prove that, for all natural numbers $n, \ell \in \mathbb{N}$, the more general identity $\vec{\nabla}\times(f^n\vec{\nabla}(f^\ell)) = \vec{\mathbf{0}}$ holds true.
(Here f^n and f^ℓ simply denote the nth and ℓth powers of f.)
Hint: use an appropriate version of the chain rule to compute the gradients of f^ℓ and f^n.

[6 marks]

2. The quadrilateral $R \subset \mathbb{R}^2$ is defined as

$$R = \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ \text{such that } x = (3-\eta)\xi, y = (2-\xi)\eta, \\ \text{for } 0 < \xi < 1,\ 0 < \eta < 1\big\}.$$

Compute the area of R.

[10 marks]

3. (a) Consider the triangular prism

$$D = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}},\ \text{such that } x > 0,\ y > 0,\ 0 < z < 1,\ x + y < 1\}$$

with vertices $\vec{\mathbf{0}}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}, (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}), (\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})$. Use the divergence theorem to compute the flux of the vector field $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}|\vec{\mathbf{r}}|^2$ through its boundary ∂D.
(Recall that the unit normal vector field $\hat{\boldsymbol{n}}$ is oriented outward on ∂D.)

[7 marks]

(b) Draw a sketch of the domain D. Without computing any surface integral, show that the flux of $\vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\mathbf{r}}|\vec{\mathbf{r}}|^2$ through three of the five faces of ∂D is zero.

[3 marks]

4. Consider a bounded, piecewise smooth domain $D \subset \mathbb{R}^3$, which can be subdivided in N non-overlapping subdomains $D_1, \ldots, D_N$ (i.e. $D = \bigcup_{j=1}^{N} D_j$ and $D_j \cap D_k = \emptyset$ for all $0 \le j \neq k \le N$).

Assume that in every subdomain D_j, the divergence theorem holds.

Prove that the divergence theorem holds in the whole domain D.

Hint: first consider the case $N = 2$ and then extend the result to a general N.

[10 marks]

[End of Question Paper]