Vector calculus MA2VC 2014–15 — Assignment 2 SOLUTIONS

(Exercise 1) *Consider the curve $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + t^3\hat{\boldsymbol{k}}$ for $-1 \le t \le 1$ and denote by Γ its path. Compute the line integral over Γ of the vector field $\vec{\mathbf{F}} = y^2\hat{\boldsymbol{\imath}} + 2xy\hat{\boldsymbol{\jmath}}$.*

(Version 1.) We can compute directly the line integral:

$$\frac{d\vec{\mathbf{a}}}{dt}(t) = \hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 3t^2\hat{\boldsymbol{k}},$$

$$\vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big) = \big(\vec{\mathbf{a}}_2(t)\big)^2\hat{\boldsymbol{\imath}} + 2\vec{\mathbf{a}}_1(t)\vec{\mathbf{a}}_2(t)\hat{\boldsymbol{\jmath}} = (t^2)^2\hat{\boldsymbol{\imath}} + 2t\,t^2\hat{\boldsymbol{\jmath}} = t^4\hat{\boldsymbol{\imath}} + 2t^3\hat{\boldsymbol{\jmath}},$$

$$\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{-1}^1 \vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t)\,\mathrm{d}t = \int_{-1}^1 (t^4\hat{\boldsymbol{\imath}} + 2t^3\hat{\boldsymbol{\jmath}}) \cdot (\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 3t^2\hat{\boldsymbol{k}})\,\mathrm{d}t = \int_{-1}^1 5t^4\,\mathrm{d}t = 2.$$

(Version 2.) We can also verify that $\varphi = xy^2$ is a scalar potential of $\vec{\mathbf{F}}$, i.e. $\vec{\nabla}\varphi = \vec{\mathbf{F}}$, and use the fundamental theorem of vector calculus (45):

$$\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi\big(\vec{\mathbf{a}}(1)\big) - \varphi\big(\vec{\mathbf{a}}(-1)\big) = \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 1 - (-1) = 2.$$

(Exercise 2) *Consider the following four curves:*

$$\begin{aligned}
&\vec{\mathbf{a}}_A(t) = \sin t\,\hat{\boldsymbol{\imath}} + \cos 44t\,\hat{\boldsymbol{\jmath}} + \sin 5t\,\hat{\boldsymbol{k}} && -\frac{\pi}{2} \le t \le \frac{\pi}{2},\\
&\vec{\mathbf{a}}_B(t) = (t+1)\hat{\boldsymbol{\imath}} + (t+1)^2\hat{\boldsymbol{\jmath}} + (t+1)^3\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
&\vec{\mathbf{a}}_C(t) = t^4\hat{\boldsymbol{\imath}} - t^{12}\hat{\boldsymbol{\jmath}} - t^2\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
&\vec{\mathbf{a}}_D(t) = \Big(\frac{4}{1+t} - 3\Big)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \frac{1}{1+t(1-t)}\hat{\boldsymbol{\jmath}} && 0 \le t \le 1,
\end{aligned}$$

Denote by $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D the corresponding paths. The integrals of the field $\vec{\mathbf{F}}$ defined above over $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D give the following values, listed in ascending order: $-2, 0, 2, 32$. Associate to every curve the value of the corresponding line integral. Justify your answer.

We first note that the field $\boxed{\vec{\mathbf{F}} \text{ is conservative}}$ on the whole of $\mathbb{R}^3$: this can be verified either by computing the scalar potential $\varphi = xy^2$, or by verifying that $\vec{\mathbf{F}}$ is irrotational ($\vec{\nabla} \times \vec{\mathbf{F}} = (2y - 2y)\hat{\boldsymbol{k}} = \vec{\mathbf{0}}$) *and* that $\vec{\mathbf{F}}$ is defined in the whole of $\mathbb{R}^3$ (see Remark 1.58).

Since $\vec{\mathbf{F}}$ is conservative, by the fundamental theorem of calculus 2.12, the value of its line integral along a certain path depends only on the value of φ at the path's endpoints. We compute the endpoints of all paths:

Γ_A	has endpoints	$\vec{\mathbf{a}}_A(-\pi/2) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_A(\pi/2) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$;
Γ_B	has endpoints	$\vec{\mathbf{a}}_B(-1) = \vec{\mathbf{0}}$,	$\vec{\mathbf{a}}_B(1) = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}$;
Γ_C	has endpoints	$\vec{\mathbf{a}}_C(-1) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_C(1) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$;
Γ_D	has endpoints	$\vec{\mathbf{a}}_D(0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_D(1) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$.

Now we can use the endpoints in two ways.

(Version 1.) From Theorem 2.16 we deduce:

- since Γ_A has the same endpoints of the path Γ of Exercise 1, the corresponding line integrals coincide;
- Γ_C is a loop because $\vec{\mathbf{a}}_C(-1) = \vec{\mathbf{a}}_C(1)$, thus by Theorem 2.16 its line integral is zero;
- Γ_D has the same endpoints of the path Γ of Exercise 1 but the order of the endpoints (thus the orientation) is reversed, so the corresponding line integrals have the same absolute value and opposite signs;
- the endpoints of Γ_B are not related to those of Γ, so we cannot immediately deduce the value of the corresponding line integral.

We conclude:

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 2, \qquad \int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = -\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = -2,$$

and the remaining value in the list gives the remaining integral $\int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 32$ (we can easily verify this value using the scalar potential and the fundamental theorem of vector calculus or computing directly the line integral).

(Version 2.) We can also compute the integrals by using the fundamental theorem of vector calculus 2.12 and the scalar potential $\varphi = xy^2$:

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 2, \qquad \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}) - \varphi(\vec{\mathbf{0}}) = 32,$$

$$\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - \varphi(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 0, \qquad \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) = -2.$$

(Exercise 3) *Compute the double integral of the field $f = \frac{x+y}{x}$ over the region*

$$R = \Big\{ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 2, 0 < \frac{1}{2}\Big(\frac{y}{x} + 1\Big) < 1 \Big\}.$$

(Version 1.) We note that R is defined by the fact that two quantities (x and $\frac{1}{2}(\frac{y}{x}+1)$), depending on x and y, belong to two intervals ($(0,2)$ and $(0,1)$, respectively). Denoting by ξ and η these two functions of x and y, we obtain the change of variables

$$\begin{cases} \xi = x, \\ \eta = \dfrac{1}{2}\Big(\dfrac{y}{x} + 1\Big), \end{cases}$$

which maps R into the rectangle $\vec{\mathbf{T}}(R) = \{\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}},\ 0 < \xi < 2, 0 < \eta < 1\}$. The inverse change of variables is:

$$\begin{cases} x = \xi, \\ y = \xi(2\eta - 1), \end{cases}$$

from which we obtain the Jacobian determinant

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \det\begin{pmatrix} 1 & 0 \\ 2\eta - 1 & 2\xi \end{pmatrix} = 2\xi.$$

The integral is computed using the change of variables formula (53):

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\vec{\mathbf{T}}(R)} \frac{x(\xi,\eta) + y(\xi,\eta)}{x(\xi,\eta)} \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta = \int_0^2 \int_0^1 2\eta\, 2\xi\,\mathrm{d}\eta\,\mathrm{d}\xi = \eta^2\Big|_{\eta=0}^1 \xi^2\Big|_{\xi=0}^2 = 4.$$

(Many other changes of variables are possible, the one above seems to be the simplest. The change of variables for y-simple domains proposed in example 2.30 in the notes coincides with this one.)

(Version 2.) Alternatively, one can use the equivalences

$$0 < \frac{1}{2}\Big(\frac{y}{x} + 1\Big) < 1 \quad \iff \quad 0 < \frac{y+x}{x} < 2 \quad \iff \quad 0 < y + x < 2x \quad \iff \quad -x < y < x$$

to rewrite the domain as $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 2, -x < y < x\}$ (from this expression it is clear that R is the triangle with vertices $\vec{\mathbf{0}}$, $2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ and $2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$) and use a simple iterated integral:

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_{-x}^x \frac{x+y}{x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \Big(y + \frac{y^2}{2x}\Big)\Big|_{y=-x}^x \mathrm{d}x = \int_0^2 \Big(x + \frac{x}{2}\Big) - \Big(-x + \frac{x}{2}\Big)\,\mathrm{d}x = \int_0^2 2x\,\mathrm{d}x = 4.$$

(Exercise 4) *Consider the following subsets of* $\mathbb{R}^3$ (T, U, V, W, X, Y, Z)*. Which of these are:*
(i) the path of a curve,
(ii) the path of a loop,
(iii) a two-dimensional (flat) region,
(iv) a graph surface,
(v) the boundary of a domain,
(vi) the level set of a scalar field defined on $\mathbb{R}^3$*,*
(vii) an oriented surface,
(viii) a domain?

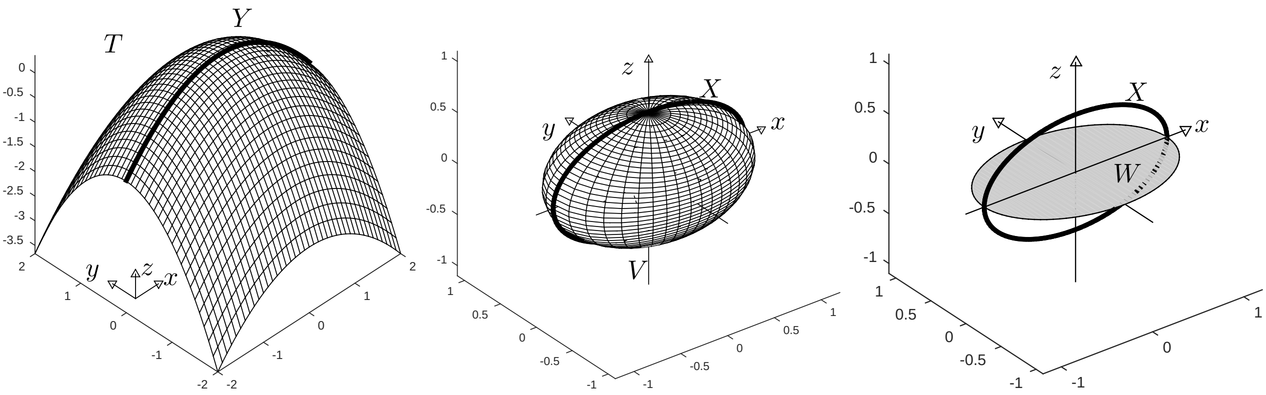

Figure 1: The sets considered in Exercise 4.
In the left plot the graph surface (defined on the unbounded region $R = \mathbb{R}^2$) is T. Its intersection with the xz-plane (the thick line) is the path Y.
In the centre plot the surface of the ellipsoid is V; the bounded domain U is the portion of space enclosed by V and the unbounded domain Z is the portion of space outside V.
In the right plot, the ellipse in the xy-plane is W; the elliptical loop in the xz-plane is X. Moreover W is the intersection of U with the xy-plane and X is the intersection of V with the xz-plane.

To classify these sets we first need to understand their shapes. They are all plotted in Figure 1.
The first classification is in term of dimensions:

- X and Y are the only 1-dimensional objects in the list, so they are **paths** of some curves, while all the other sets are not.
  Are these curves also **loops**?
  - X is an ellipse, so it is the path of a loop because any parametrisation of it must end in the same point it starts from.
  - On the other hand, Y is a parabola, so it is not a loop (it crosses every plane parallel to the yz-plane only once, so it can not "return" to its starting point).
- U and Z are the only three-dimensional sets in the list. Since they are defined through strict inequalities only, they are open sets, i.e. **domains**.
  U is the portion of space inside the ellipsoid V, and Z is the part of space outside it.
- T, V and W are two dimensional sets, i.e. **surfaces**. In particular they are all oriented surfaces. Moreover:
  - Only W is a two-dimensional **region**, as it lies in the xy-plane.
    T and V are curved, so are not subset of any plane.
  - T and W are **graph surfaces**: in both cases the variable z can be expressed as function of the variables x and y. For T we have $z = \frac{1}{3}(1 - x^2 - y^2)$, while for W we simply have $z = 0$.
    On the other hand, V is not a graph as the vertical lines crossing V cross it in two points (e.g. the two points $\pm 1/\sqrt{3}\hat{\boldsymbol{k}}$ belongs to V and lie on the same vertical line). In other words, V is the union of two graphs defined on the same region W: the graphs of $g_{V+} = \sqrt{(1 - x^2 - 2y^2)/3}$ and of $g_{V-} = -\sqrt{(1 - x^2 - 2y^2)/3}$.
  - T and V are **boundaries**: V separates the ellipsoid U from the exterior domain Z, so it is boundary of both ($V = \partial U = \partial Z$); T is an unbounded surface separating the part of $\mathbb{R}^3$ under it from the part above it.
    W is not boundary of any domain, as it does not separate the space in two parts.

What about **level sets**? The surfaces T and W are already written as level sets for continuous scalar fields. The domains U and Z and the region W can not be written as level sets of any continuous field (this is related to the fact they are defined using strict inequalities). The paths X and Y can be written as level sets (see table below), even if in all the examples we have seen in class the level sets were surfaces (the issue here is that they are level sets for scalar field whose gradient vanishes on them).

Note that every set has a precise dimension (1, 2, or 3), so it is either a path, or a surface or a domain, and cannot be classified as two of these objects simultaneously.

We summarise the discussion above in the following table.

$T = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z = 1\}$,	(iv) graph surface of $g_T = \frac{1}{3}(1 - x^2 - 2y^2)$ on $R = \mathbb{R}^2$, (v) boundary of the (unbounded) domain $\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z < 1\}$, (vi) level set of $f_T = x^2 + 2y^2 + 3z$ for $\lambda = 1$, (vii) oriented surface ($\hat{\boldsymbol{n}} = \vec{\nabla} f_T / \vert\vec{\nabla} f_T\vert$),
$U = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z^2 < 1\}$,	(viii) domain (defined by inequalities only),
$V = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z^2 = 1\}$,	(v) boundary of U, (vi) level set of $f_V = x^2 + 2y^2 + 3z^2$ for $\lambda = 1$, (vii) oriented surface ($\hat{\boldsymbol{n}} = \vec{\nabla} f_V / \vert\vec{\nabla} f_V\vert$),
$W = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z^2 < 1,\ z = 0\}$,	(iii) 2D region, (iv) graph of constant field $g = 0$ on $R = W$, (vii) oriented surface, W can be considered also as a "2D domain" (viii),
$X = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z^2 = 1,\ y = 0\}$,	(i) path of curve $\vec{\mathbf{a}}_X(t) = \cos t\, \hat{\boldsymbol{\imath}} + \frac{1}{\sqrt{3}} \sin t\, \hat{\boldsymbol{k}},\ t \in [0, 2\pi]$, (ii) $\vec{\mathbf{a}}_X$ is a loop because $\vec{\mathbf{a}}_X(0) = \vec{\mathbf{a}}_X(2\pi)$ (it is an ellipse), (vi) level set of $f_X = (x^2 + 2y^2 + 3z^2 - 1)^2 + y^2$ for $\lambda = 0$,
$Y = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z = 1,\ y = 0\}$,	(i) path of curve $\vec{\mathbf{a}}_Y(t) = t\hat{\boldsymbol{\imath}} + \frac{1}{3}(1 - t^2)\hat{\boldsymbol{k}},\ t \in \mathbb{R}$, (vi) level set of $f_Y = (x^2 + 2y^2 + 3z - 1)^2 + y^2$ for $\lambda = 0$,
$Z = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x^2 + 2y^2 + 3z^2 > 1\}$,	(viii) domain (defined by inequalities only).

In most cases, a good heuristic way to guess the dimension of a subset of $\mathbb{R}^3$ from its analytical expression is to count the number of *equations* in its definition: if there is none, then the set is three dimensional (e.g. is a domain); if there is only one equation, then the set is two-dimensional (a surface); if there are two then the set is one-dimensional (a path). The number of strict *inequalities* does not affect the dimension of the set (in most cases this is true also for $\leq$ and $\geq$ inequalities, but not always, consider for example the sets $\{x^2 \leq 0\}, \{x^2 + y^2 \leq 0\}, \{x^2 + y^2 + z^2 \leq 0\}$). However, there are some cases where this technique does not apply: for example the path X (defined above with two equations, as expected) can also be written as $X = \{\vec{\mathbf{r}} \in \mathbb{R}^3, (x^2 + 2y^2 + 3z^2 - 1)^2 + y^2 = 0\}$).