Vector calculus MA2VC 2016–17: Assignment 2
SOLUTIONS AND FEEDBACK

(Exercise 1) *Consider the following (planar) curves:*

$$\begin{aligned}
\vec{\mathbf{a}}(t) &= -\mathrm{e}^t\hat{\boldsymbol{\imath}} + \mathrm{e}^t\hat{\boldsymbol{\jmath}} && -\infty < t \le 0,\\
\vec{\mathbf{b}}(t) &= \sqrt{3}t^2\hat{\boldsymbol{\imath}} + (t^3 - t)\hat{\boldsymbol{\jmath}} && -1 \le t \le 1,\\
\vec{\mathbf{c}}(t) &= \cos t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}} && 0 \le t \le \pi,\\
\vec{\mathbf{d}}(t) &= -t\hat{\boldsymbol{\imath}} + \Big(1 - \frac{(t+1)^{3/2}}{\sqrt{2}}\Big)\hat{\boldsymbol{\jmath}} && -1 \le t \le 1.
\end{aligned}$$

Denote by $\Gamma_a, \Gamma_b, \Gamma_c$ and Γ_d the paths of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}, \vec{\mathbf{d}}$, respectively.

(i) Which of these curves are loops?

(ii) Two of these curves share the same endpoints, which ones?

(iii) Which of the four curves has the shortest path? Which the longest?

(iv) Compute the line integral of the scalar field $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^2$ along the path Γ_a.

(v) Compute the line integral of the vector field $x^2\hat{\boldsymbol{\imath}} + xy\hat{\boldsymbol{\jmath}}$ along the path Γ_c.

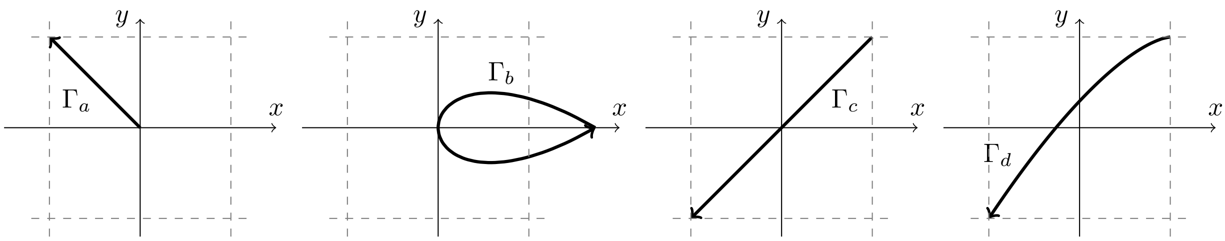
(vi) Without computing any integral, show that

$$\int_{\Gamma_b} (x^{10}\hat{\boldsymbol{\imath}} + y^{10}\hat{\boldsymbol{\jmath}}) \cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_d} \vec{\mathbf{r}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_c} (\cosh x \tan y\hat{\boldsymbol{\imath}} - \cosh x \tan y\hat{\boldsymbol{\jmath}}) \cdot \mathrm{d}\vec{\mathbf{r}} = 0.$$

(vii) Find a conservative vector field $\vec{\mathbf{F}}$ such that its line integral over each of the four paths is 1, or prove that no such field exists.

Justify all your answers.
Hint: *Sketch the paths of the curves and use definition and results available in the lecture notes.*

We first show the four paths (the dashed lines represents the lines $x = \pm 1$ and $y = \pm 1$).



(i) We compute all endpoints of the curves:

$$\begin{aligned}
\lim_{t\to-\infty} \vec{\mathbf{a}}(t) &= \vec{\mathbf{0}}, \qquad \vec{\mathbf{a}}(0) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}},\\
\vec{\mathbf{b}}(-1) &= \vec{\mathbf{b}}(1) = \sqrt{3}\hat{\boldsymbol{\imath}},\\
\vec{\mathbf{c}}(0) &= \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{c}}(\pi) = -\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}},\\
\vec{\mathbf{d}}(-1) &= \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, \qquad \vec{\mathbf{d}}(1) = -\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}}.
\end{aligned}$$

The only curve whose initial and final endpoints are equal is $\boxed{\vec{\mathbf{b}}}$, so this is the only loop.
(Moreover, curve $\vec{\mathbf{a}}$ is defined on an unbounded interval, so it cannot by a loop by our definition.)

(ii) From the computations in (i), $\vec{\mathbf{c}}(0) = \vec{\mathbf{d}}(-1)$ and $\vec{\mathbf{c}}(\pi) = \vec{\mathbf{d}}(1)$, i.e. the endpoints of $\boxed{\Gamma_c \text{ and } \Gamma_d}$ coincide.

(iii) We compute the lengths of the four paths as the line integral of the constant field 1 (recall equation (43)):

$$\begin{aligned}
\frac{d\vec{\mathbf{a}}}{dt}(t) &= -\mathrm{e}^t\hat{\boldsymbol{\imath}} + \mathrm{e}^t\hat{\boldsymbol{\jmath}}, & \left|\frac{d\vec{\mathbf{a}}}{dt}(t)\right| &= \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{2t}} = \sqrt{2}\mathrm{e}^t,\\
\frac{d\vec{\mathbf{b}}}{dt}(t) &= 2\sqrt{3}t\hat{\boldsymbol{\imath}} + (3t^2 - 1)\hat{\boldsymbol{\jmath}}, & \left|\frac{d\vec{\mathbf{b}}}{dt}(t)\right| &= \sqrt{12t^2 + 9t^4 - 6t^2 + 1} = 3t^2 + 1,
\end{aligned}$$

$$\frac{d\vec{\mathbf{c}}}{dt}(t) = -\sin t\hat{\boldsymbol\imath} - \sin t\hat{\boldsymbol\jmath}, \qquad \left|\frac{d\vec{\mathbf{c}}}{dt}(t)\right| = \sqrt{2}\sin t,$$

$$\frac{d\vec{\mathbf{d}}}{dt}(t) = -\hat{\boldsymbol\imath} + -\frac{3(t+1)^{1/2}}{2\sqrt{2}}\hat{\boldsymbol\jmath}, \qquad \left|\frac{d\vec{\mathbf{d}}}{dt}(t)\right| = \sqrt{1+\frac{9}{8}(t+1)} = \sqrt{\frac{17}{8}+\frac{9}{8}t},$$

$$\begin{aligned}
\mathrm{Length}(\Gamma_a) &= \int_{\Gamma_a} 1\,\mathrm{d}s = \int_{-\infty}^{0} \sqrt{2}\mathrm{e}^t\,\mathrm{d}t = \sqrt{2}\mathrm{e}^t\Big|_{-\infty}^{0} = \sqrt{2},\\
\mathrm{Length}(\Gamma_b) &= \int_{\Gamma_b} 1\,\mathrm{d}s = \int_{-1}^{1} (3t^2+1)\,\mathrm{d}t = (t^3+t)\Big|_{-1}^{1} = 4,\\
\mathrm{Length}(\Gamma_c) &= \int_{\Gamma_c} 1\,\mathrm{d}s = \int_{0}^{\pi} \sqrt{2}\sin t\,\mathrm{d}t = -\sqrt{2}\cos t\Big|_{0}^{\pi} = 2\sqrt{2},\\
\mathrm{Length}(\Gamma_d) &= \int_{\Gamma_d} 1\,\mathrm{d}s = \int_{-1}^{1} \sqrt{\frac{17}{8}+\frac{9}{8}t}\,\mathrm{d}t = \frac{16}{27}\left(\frac{17}{8}+\frac{9}{8}t\right)^{3/2}\Big|_{-1}^{1} = \frac{16}{27}\Big((13/4)^{3/2}-1\Big) \approx 2.879.
\end{aligned}$$

Thus $\boxed{\text{the shortest path is } \Gamma_a \text{ and the longest one is } \Gamma_b}$. [1]

(iv)

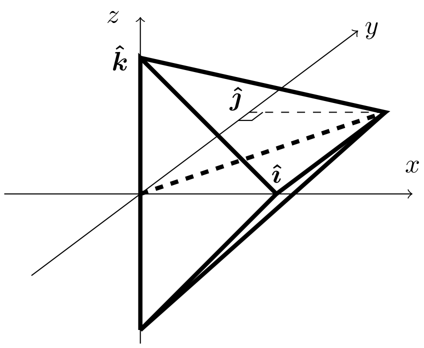
$$\int_{\Gamma_a} |\vec{\mathbf{r}}|^2\,\mathrm{d}s \overset{(41)}{=} \int_{-\infty}^{0} |\vec{\mathbf{a}}(t)|^2 \left|\frac{d\vec{\mathbf{a}}}{dt}(t)\right|\mathrm{d}t = \int_{-\infty}^{0} 2\mathrm{e}^{2t}\sqrt{2}\mathrm{e}^t\,\mathrm{d}t = \frac{2\sqrt{2}}{3}\mathrm{e}^{3t}\Big|_{-\infty}^{0} = \boxed{\frac{2\sqrt{2}}{3}} \approx 0.9428.$$

(v)

$$\int_{\Gamma_a} (x^2\hat{\boldsymbol\imath} + xy\hat{\boldsymbol\jmath})\cdot \mathrm{d}\vec{\mathbf{r}} \overset{(44)}{=} \int_0^\pi (\cos^2 t\hat{\boldsymbol\imath} + \cos^2 t\hat{\boldsymbol\jmath})\cdot(-\sin t\hat{\boldsymbol\imath} - \sin t\hat{\boldsymbol\jmath})\,\mathrm{d}t = -2\int_0^\pi \cos^2 t\sin t\,\mathrm{d}t = \frac{2}{3}\cos^3 t\Big|_0^\pi = \boxed{-\frac{4}{3}}.$$

(vi)
- Since $\vec{\mathbf{F}}(\vec{\mathbf{r}}) := x^{10}\hat{\boldsymbol\imath} + y^{10}\hat{\boldsymbol\jmath} = \nabla(\frac{x^{11}+y^{11}}{11})$, the field $\vec{\mathbf{F}}$ is conservative.[2]
  By Theorem 2.19, since $\boxed{\vec{\mathbf{F}} \text{ is conservative and } \vec{\mathbf{b}} \text{ is a loop}}$, $\int_{\Gamma_b} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = 0$.
- The position vector $\vec{\mathbf{r}}$ is conservative with scalar potential $\boxed{\phi(\vec{\mathbf{r}}) = \frac{1}{2}|\vec{\mathbf{r}}|^2}$.
  Thus $\int_{\Gamma_d} \vec{\mathbf{r}}\cdot \mathrm{d}\vec{\mathbf{r}} = \phi(-\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath}) - \phi(\hat{\boldsymbol\imath} + \hat{\boldsymbol\jmath}) = 1 - 1 = 0$.
- The field $(\cosh x \tan y)(\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})$ is not conservative, thus for the last integral we cannot use the same argument. However, the field $(\cosh x\tan y)(\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})$ points in the direction $\pm(\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})$, so it is $\boxed{\text{perpendicular to the segment } \Gamma_c}$ which has unit tangent vector $(-\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})/\sqrt{2}$. Since the line integral of a vector field is the integral of its component *parallel* to the path (recall the explanation on page 36 of the notes and equation (45) in particular), its integral is zero.[3]

(vii) By Theorem 2.19, if a field $\vec{\mathbf{F}}$ is conservative then its line integral over any loop is zero. Since Γ_b is a loop, $\int_{\Gamma_b} \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = 0 \neq 1$ thus $\boxed{\text{no such field exists}}$. [4]



[1] Since Γ_a and Γ_c are straight segments we could have computed their lengths without integrals: $\mathrm{Length}(\Gamma_a) = |\vec{\mathbf{a}}(0) - \vec{\mathbf{a}}(\infty)| = |-\hat{\boldsymbol\imath} + \hat{\boldsymbol\jmath}|$ and $\mathrm{Length}(\Gamma_c) = |\vec{\mathbf{a}}(\pi) - \vec{\mathbf{a}}(0)| = |-2\hat{\boldsymbol\imath} - 2\hat{\boldsymbol\jmath}|$. Given the endpoints of the paths and that $\vec{\mathbf{b}}(0) = \vec{\mathbf{0}}$ (thus $\mathrm{Length}(\Gamma_b) \geq 2\sqrt{3}$ and $\mathrm{Length}(\Gamma_d) \geq 2\sqrt{2}$), we have that Γ_a is the shortest path; however to verify Γ_d is not longer than Γ_b we need to compute explicitly its length, as we did above.

[2] Equivalently, $\nabla\times\vec{\mathbf{F}} = \vec{\mathbf{0}}$, so $\vec{\mathbf{F}}$ is irrotational. Being defined over all $\mathbb{R}^3$ (which is a star-shaped set), by Theorem 2.18 $\vec{\mathbf{F}}$ is also conservative.

[3] One could also verify that $(\cosh x\tan y)(\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})\cdot\frac{d\vec{\mathbf{c}}}{dt}(t) = (\cosh\cos t)(\tan\cos t)(\sin t)(\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath})\cdot(-\hat{\boldsymbol\imath} - \hat{\boldsymbol\jmath}) = 0$.

[4] It is easy to write a planar conservative field $\vec{\mathbf{G}}$ such that $\int_{\Gamma_a}\vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_c}\vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_d}\vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{r}} = 1$, for example $\vec{\mathbf{G}} = \vec{\nabla}(\frac{1}{4}y - \frac{3}{4}x) = -\frac{3}{4}\hat{\boldsymbol\imath} + \frac{1}{4}\hat{\boldsymbol\jmath}$. So the only obstacle to the existence of the desired field is the loop Γ_b.

(Exercise 2) *Compute the triple integral $\iiint_D f\,\mathrm{d}V$, where $f(\vec{\mathbf{r}}) = xy$ and D is the tetrahedron*

$$D = \{\vec{\mathbf{r}} \in \mathbb{R}^3, \quad 0 < y < x, \quad |z| < 1 - x\}$$

(depicted in the figure).
Hint: *Write D as a z-simple domain.*

We need to write the extremes of integrations for the variables x, y, z in a way suitable for the computation of the triple integral. This means that one of the variables (that in the outer integral) has to be bounded independently of the other two, while the extremes for the other two can depend on the first one.

The variable x is positive and satisfies $x < 1 - |z|$, so in particular $0 < x < 1$ (as we see in the figure, D lies between the planes $\{x = 0\}$ and $\{x = 1\}$). The variable y satisfies $0 < y < x$ (equivalently, D lies between the planes $\{y = 0\}$ and $\{y = x\}$). The condition $|z| < 1 - x$ gives $x - 1 < z < 1 - x$ (again, D lies between the planes $\{z = x - 1\}$ and $\{z = 1 - x\}$).

Equivalently, one can note that D agrees with the definition of a z-simple domain as on page 52, with $x_L = 0$, $x_R = 1$, $a(x) = 0$, $b(x) = x$, $\alpha(x, y) = x - 1$, $\beta(x, y) = 1 - x$.

Using formula (62) for the triple integral on a z-simple domain,

$$\begin{aligned}\iiint_D f\,\mathrm{d}V &= \int_0^1 \left(\int_0^x \left(\int_{x-1}^{1-x} xy\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 \int_0^x xy2(1-x)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x2(1-x)\frac{y^2}{2}\bigg|_{y=0}^{x} \mathrm{d}x = \int_0^1 x^3(1-x)\,\mathrm{d}x = \frac{x^4}{4} - \frac{x^5}{5}\bigg|_0^1 = \boxed{\frac{1}{20}}.\end{aligned}$$

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MA2VC: Feedback after grading assignment 2

Check carefully the points below and all the corrections in your assignment; even if you got full marks, your solution can probably be improved. See also page 110 in the notes.

In your coursework, a red check mark ✓ means "correct", a × mark means "error", a check mark in brackets (✓) means "correct step leading to a wrong solution due to previous errors".

If you have any question or comment about the assignment, the solutions or the marking, please do ask me.

Exercise 1: Parts from (i) to (v) were straightforward and most students did well. Part (vi) required to apply slightly abstract result to the computation of integrals, as opposed to the direct application of formulas: many students showed very poor understanding of line integrals and in particular of the fundamental theorem of vector calculus and its consequences.

If you didn't get full marks (or if you got them but you are not sure about your answer), please revise carefully the comments below and section 2.1.3 in the notes.

(i) In part (i), since you are asked to justify your answer, you have to state why paths $\Gamma_a, \Gamma_c, \Gamma_d$ are not loops.

(i) In (i), you can say that $\vec{\mathbf{a}}$ is not a loop because it is defined on the interval $(-\infty, 0]$, which is not bounded, contradicting our definition of loop (page 10). However, $(-\infty, 0]$ *is* closed, as you should know from last year. Moreover, saying that "$\vec{\mathbf{a}}$ is not bounded" is not correct: the interval of definition of $\vec{\mathbf{a}}$ is not bounded, not $\vec{\mathbf{a}}$ itself.

(i) Many students computed the endpoints of a path, then checked that one of the components of the parametrisation is strictly decreasing. What's the point of this? If $\vec{\mathbf{c}}(0) \neq \vec{\mathbf{c}}(\pi)$, then Γ_c is not a loop, and there is no need to do further computations.

(ii) The interval $[t_1, t_2] \subset \mathbb{R}$ has endpoints t_1 and t_2. In the same way, the path Γ_a has endpoints $\vec{\mathbf{0}}$ and $-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$: each path has *two* endpoints, a starting one and a final one. Not only the final point is an endpoint, but also the initial one. This is important, because it is the meaning of the word "endpoint" used e.g. in the fundamental theorem of vector calculus or in the statement of Theorem 2.19.

(iii) In part (iii) you could have computed the length of Γ_a and Γ_c simply as the distance between the endpoints, with no need of integrals, since these paths are straight segments.

(iii) No path can have length 0, if you find such a value you must realise you have done some error.

(v) Solving part (v), one encounters the integral $\int_0^\pi -2\cos^2 t \sin t\,\mathrm{d}t$. Since $\cos' t = -\sin t$, the integrand is clearly the derivative of $\frac{2}{3}\cos^3 t$. Most students used integration by substitutions and this led several to (sign) errors.

(vi.1) Regarding the first integral of part (vi), plenty of students wrote things like: "By Theorem 2.19, the integral of a vector field over a loop is zero, so...". Theorem 2.19 does not say this. It states that for a given vector field $\vec{\mathbf{F}}$, all the integrals of $\vec{\mathbf{F}}$ over loops are zero *if and only if* $\vec{\mathbf{F}}$ is conservative.
The word "equivalent" in the statement means that one condition is true if and only if the other is true.
For example, the integral of $x\hat{\boldsymbol{\jmath}}$ over the unit circle $\{|\vec{\mathbf{r}}| = 1, z = 0\}$ is π, which is not 0.
So here you must verify that $\vec{\mathbf{F}} = x^{10}\hat{\boldsymbol{\imath}} + y^{10}\hat{\boldsymbol{\jmath}}$ is conservative.

(vi.1) Conservative implies irrotational, but $\boxed{\text{irrotational does not imply conservative}}$.

Recall the field $\frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$, which is irrotational (verify it!). However its line integral along the unit circle $\{|\vec{\mathbf{r}}|=1, z=0\}$ is 2π (verify it!), so by Theorem 2.19, this field is not conservative.

When dealing with the first integral of part (vi), denoting $\vec{\mathbf{F}} = x^{10}\hat{\imath} + y^{10}\hat{\jmath}$, plenty of students wrote "$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ so $\vec{\mathbf{F}}$ is conservative". This is not correct. $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ implies only that $\vec{\mathbf{F}}$ is irrotational (since $\vec{\nabla} \times \vec{\mathbf{F}} = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\hat{\boldsymbol{k}}$ for planar fields). To deduce that $\vec{\mathbf{F}}$ is conservative, one has to note also that the field $\vec{\mathbf{F}} = x^{10}\hat{\imath} + y^{10}\hat{\jmath}$ is defined on the whole of $\mathbb{R}^3$ (which is star-shaped) and apply Theorem 2.18.

So in order to verify that the first integral in part (vi) is zero one has to:
(A) use Theorem 2.19, and
(B) prove that $\vec{\mathbf{F}}$ is conservative.
To prove (B), one can either
(B.1) compute a scalar potential of $\vec{\mathbf{F}}$, or
(B.2) verify that $\vec{\mathbf{F}}$ is irrotational AND that is defined on the whole of $\mathbb{R}^3$.
In physics and engineering books you might find that a planar field is conservative if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. In this cases there is an underlying assumption that the field is defined on the whole plane (or space), otherwise this statement is false: recall the field $\frac{-y\hat{\imath}+x\hat{\jmath}}{x^2+y^2}$ mentioned above (which is not defined for $x = y = 0$). Since we never made this assumption, you have to verify it for the field at hand.

(vi.1) In the answer to part (vi), several students found the correct scalar potential for the first integrand, and denoted it $\phi = \frac{1}{11}(x^{11} + y^{11})$. Then they wrote that $\int_{\Gamma_b}(x^{10}\hat{\imath} + x^{11}\hat{\jmath}) \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_b} \mathrm{d}\phi = 0$. We never used the notation $\mathrm{d}\phi$. When you introduce some notation that is not used in the lectures, you must explain precisely what it means.

(vi) Many people tried to prove that $\int_{\Gamma_d} \vec{\mathbf{r}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ and $\int_{\Gamma_c}(\cosh x \tan y\hat{\imath} - \cosh x \tan y\hat{\jmath}) \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ using the fact that the paths Γ_c and Γ_d have the same endpoints. This does not make sense. Two integrals on two paths with common endpoints are equal (but not necessarily equal to 0) if they are integrals of the same conservative vector field (by Theorem 2.19 again). Here the integrands $\vec{\mathbf{r}}$ and $\cosh x \tan y\hat{\imath} - \cosh x \tan y\hat{\jmath}$ are different (and the second is not even conservative), so their integrals are not related.
(If this was true, i.e. integrals of all possible fields over a given path are equal, then all line integrals would equal the line integral of the trivial field $\vec{\mathbf{F}} = \vec{\mathbf{0}}$. So all path integrals would be zero, which would make their study completely pointless.)

(vii) Some students wrote that the field asked for in question (vii) does not exist because conservative fields integrated on paths with different endpoints must give different values. In formulas, since $\int_{\Gamma_a} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\vec{\mathbf{0}}}^{-\hat{\imath}+\hat{\jmath}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ and $\int_{\Gamma_c} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\hat{\imath}+\hat{\jmath}}^{-\hat{\imath}-\hat{\jmath}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ then these integrals have to differ because the endpoints are different. This is false: a simple counterexample is $\vec{\mathbf{G}} = \vec{\nabla}(\frac{1}{4}y - \frac{3}{4}x) = -\frac{3}{4}\hat{\imath} + \frac{1}{4}\hat{\jmath}$, for which $\int_{\Gamma_a} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_d} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = 1$.
It all boils down once again to "A implies B does not give that B implies A": the integrals of a conservative field on two paths with common endpoints are equal, but the integrals of the same field on two paths with different endpoints do not need to be different.

- Try to write precise and clear sentences.
  For example you cannot say things like "path Γ is 0", when you mean that some integral over Γ is 0.
  In particular, do not mix up paths (geometric objects, the subsets of $\mathbb{R}^3$ *on* which we compute line integrals) and fields (the integrands *of* which we compute integrals). So, if you say "I integrate this path", you commit an error: you can integrate fields on paths, but you cannot integrate paths.

Exercise 2: Most people solved exercise 2 correctly.

The only difficulty was in finding the extremes of integration for the x variable, as those for y and z were given in the question.
The lower bound $x > 0$ follows simply from $x > y > 0$.
The upper bound $x < 1$ follows from $|z| < 1 - x \Rightarrow x < 1 - |z| < 1$, since $|z|$ is non-negative.
These could have been deduced from the figure: the shape D lies between the planes $x = 0$ and $x = 1$.

If you want to know how to draw the figure of D from its definition, you can first see that D is defined by four linear inequalities $D = \{0 < y,\ y < x,\ z < 1-x,\ x-1 < z\}$. So D is the intersection of four half-spaces, i.e. a tetrahedron. Then the vertices of the tetrahedron are simply the points where three of these four inequalities become equality:

$$\{\hat{\imath}\} = \{0 = y,\ y < x,\ z = 1-x,\ x-1 = z\}, \quad \{\hat{\boldsymbol{k}}\} = \{0 = y,\ y = x,\ z = 1-x,\ x-1 < z\},$$
$$\{-\hat{\boldsymbol{k}}\} = \{0 = y,\ y = x,\ z < 1-x,\ x-1 = z\}, \quad \{\hat{\imath}+\hat{\jmath}\} = \{0 < y,\ y = x,\ z = 1-x,\ x-1 = z\}.$$

Then it suffices to connect the dots.
Alternatively, one can draw the four planes corresponding to the four inequalities.