Vector calculus MA3VC 2014–15 — Assignment 2
SOLUTIONS

(Exercise 1) *Consider the curve $\vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}} + t^3\hat{\boldsymbol{k}}$ for $-1 \le t \le 1$ and denote by Γ its path. Compute the line integral over Γ of the vector field $\vec{\mathbf{F}} = y^2\hat{\boldsymbol{\imath}} + 2xy\hat{\boldsymbol{\jmath}}$.*

(Version 1.) We can compute directly the line integral:

$$\frac{d\vec{\mathbf{a}}}{dt}(t) = \hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 3t^2\hat{\boldsymbol{k}},$$

$$\vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big) = \big(\vec{\mathbf{a}}_2(t)\big)^2\hat{\boldsymbol{\imath}} + 2\vec{\mathbf{a}}_1(t)\vec{\mathbf{a}}_2(t)\hat{\boldsymbol{\jmath}} = (t^2)^2\hat{\boldsymbol{\imath}} + 2t\,t^2\hat{\boldsymbol{\jmath}} = t^4\hat{\boldsymbol{\imath}} + 2t^3\hat{\boldsymbol{\jmath}},$$

$$\int_\Gamma \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_{-1}^1 \vec{\mathbf{F}}\big(\vec{\mathbf{a}}(t)\big)\cdot\frac{d\vec{\mathbf{a}}}{dt}(t)\,\mathrm{d}t = \int_{-1}^1 (t^4\hat{\boldsymbol{\imath}} + 2t^3\hat{\boldsymbol{\jmath}})\cdot(\hat{\boldsymbol{\imath}} + 2t\hat{\boldsymbol{\jmath}} + 3t^2\hat{\boldsymbol{k}})\,\mathrm{d}t = \int_{-1}^1 5t^4\,\mathrm{d}t = 2.$$

(Version 2.) We can also verify that $\varphi = xy^2$ is a scalar potential of $\vec{\mathbf{F}}$, i.e. $\vec{\nabla}\varphi = \vec{\mathbf{F}}$, and use the fundamental theorem of vector calculus (45):

$$\int_\Gamma \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = \varphi\big(\vec{\mathbf{a}}(1)\big) - \varphi\big(\vec{\mathbf{a}}(-1)\big) = \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 1 - (-1) = 2.$$

(Exercise 2) *Consider the following four curves:*

$$\begin{aligned}
\vec{\mathbf{a}}_A(t) &= \sin t\,\hat{\boldsymbol{\imath}} + \cos 44t\,\hat{\boldsymbol{\jmath}} + \sin 5t\,\hat{\boldsymbol{k}} && -\frac{\pi}{2} \le t \le \frac{\pi}{2},\\
\vec{\mathbf{a}}_B(t) &= (t+1)\hat{\boldsymbol{\imath}} + (t+1)^2\hat{\boldsymbol{\jmath}} + (t+1)^3\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
\vec{\mathbf{a}}_C(t) &= t^4\hat{\boldsymbol{\imath}} - t^{12}\hat{\boldsymbol{\jmath}} - t^2\hat{\boldsymbol{k}} && -1 \le t \le 1,\\
\vec{\mathbf{a}}_D(t) &= \Big(\frac{4}{1+t} - 3\Big)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + \frac{1}{1+t(1-t)}\hat{\boldsymbol{\jmath}} && 0 \le t \le 1,
\end{aligned}$$

Denote by $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D the corresponding paths. The integrals of the field $\vec{\mathbf{F}}$ defined above over $\Gamma_A, \Gamma_B, \Gamma_C$ and Γ_D give the following values, listed in ascending order: $-2, 0, 2, 32$. Associate to every curve the value of the corresponding line integral. Justify your answer.

We first note that the field $\boxed{\vec{\mathbf{F}} \text{ is conservative}}$ on the whole of $\mathbb{R}^3$: this can be verified either by computing the scalar potential $\varphi = xy^2$, or by verifying that $\vec{\mathbf{F}}$ is irrotational ($\vec{\nabla}\times\vec{\mathbf{F}} = (2y - 2y)\hat{\boldsymbol{k}} = \vec{\mathbf{0}}$) *and* that $\vec{\mathbf{F}}$ is defined in the whole of $\mathbb{R}^3$ (see Remark 1.58).

Since $\vec{\mathbf{F}}$ is conservative, by the fundamental theorem of calculus 2.12, the value of its line integral along a certain path depends only on the value of φ at the path's endpoints. We compute the endpoints of all paths:

Γ_A	has endpoints	$\vec{\mathbf{a}}_A(-\pi/2) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_A(\pi/2) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$;
Γ_B	has endpoints	$\vec{\mathbf{a}}_B(-1) = \vec{\mathbf{0}}$,	$\vec{\mathbf{a}}_B(1) = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}$;
Γ_C	has endpoints	$\vec{\mathbf{a}}_C(-1) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_C(1) = \hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$;
Γ_D	has endpoints	$\vec{\mathbf{a}}_D(0) = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$,	$\vec{\mathbf{a}}_D(1) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$.

Now we can use the endpoints in two ways.

(Version 1.) From Theorem 2.16 we deduce:

- since Γ_A has the same endpoints of the path Γ of Exercise 1, the corresponding line integrals coincide;
- Γ_C is a loop because $\vec{\mathbf{a}}_C(-1) = \vec{\mathbf{a}}_C(1)$, thus by Theorem 2.16 its line integral is zero;
- Γ_D has the same endpoints of the path Γ of Exercise 1 but the order of the endpoints (thus the orientation) is reversed, so the corresponding line integrals have the same absolute value and opposite signs;
- the endpoints of Γ_B are not related to those of Γ, so we cannot immediately deduce the value of the corresponding line integral.

We conclude:

$$\int_{\Gamma_A}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = 2, \qquad \int_{\Gamma_C}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = 0, \qquad \int_{\Gamma_D}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = -\int_\Gamma \vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = -2,$$

and the remaining value in the list gives the remaining integral $\int_{\Gamma_B}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}} = 32$ (we can easily verify this value using the scalar potential and the fundamental theorem of vector calculus or computing directly the line integral).

(Version 2.) We can also compute the integrals by using the fundamental theorem of vector calculus 2.12 and the scalar potential $\varphi = xy^2$:

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) - \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 2, \qquad \int_{\Gamma_B} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}}) - \varphi(\vec{\mathbf{0}}) = 32,$$

$$\int_{\Gamma_C} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - \varphi(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) = 0, \qquad \int_{\Gamma_D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}) - \varphi(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}) = -2.$$

(Exercise 3) *Compute the double integral of the field $f = \frac{x+y}{x}$ over the region*

$$R = \Big\{ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 2, 0 < \frac{1}{2}\Big(\frac{y}{x} + 1\Big) < 1 \Big\}.$$

(Version 1.) We note that R is defined by the fact that two quantities (x and $\frac{1}{2}(\frac{y}{x} + 1)$), depending on x and y, belong to two intervals ($(0, 2)$ and $(0, 1)$, respectively). Denoting by ξ and η these two functions of x and y, we obtain the change of variables

$$\begin{cases} \xi = x, \\ \eta = \dfrac{1}{2}\Big(\dfrac{y}{x} + 1\Big), \end{cases}$$

which maps R into the rectangle $\vec{\mathbf{T}}(R) = \{\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}},\ 0 < \xi < 2, 0 < \eta < 1\}$. The inverse change of variables is:

$$\begin{cases} x = \xi, \\ y = \xi(2\eta - 1), \end{cases}$$

from which we obtain the Jacobian determinant

$$\frac{\partial(x, y)}{\partial(\xi, \eta)} = \det \begin{pmatrix} 1 & 0 \\ 2\eta - 1 & 2\xi \end{pmatrix} = 2\xi.$$

The integral is computed using the change of variables formula (53):

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{\vec{\mathbf{T}}(R)} \frac{x(\xi, \eta) + y(\xi, \eta)}{x(\xi, \eta)} \left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right| \mathrm{d}\xi\,\mathrm{d}\eta = \int_0^2 \int_0^1 2\eta\, 2\xi\,\mathrm{d}\eta\,\mathrm{d}\xi = \eta^2\Big|_{\eta=0}^{1} \xi^2\Big|_{\xi=0}^{2} = 4.$$

(Many other changes of variables are possible, the one above seems to be the simplest. The change of variables for y-simple domains proposed in example 2.30 in the notes coincides with this one.)

(Version 2.) Alternatively, one can use the equivalences

$$0 < \frac{1}{2}\Big(\frac{y}{x} + 1\Big) < 1 \quad\iff\quad 0 < \frac{y + x}{x} < 2 \quad\iff\quad 0 < y + x < 2x \quad\iff\quad -x < y < x$$

to rewrite the domain as $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 2, -x < y < x\}$ (from this expression it is clear that R is the triangle with vertices $\vec{\mathbf{0}}$, $2\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}}$ and $2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$) and use a simple iterated integral:

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_{-x}^{x} \frac{x + y}{x}\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \Big(y + \frac{y^2}{2x}\Big)\Big|_{y=-x}^{x} \mathrm{d}x = \int_0^2 \Big(x + \frac{x}{2}\Big) - \Big(-x + \frac{x}{2}\Big)\,\mathrm{d}x = \int_0^2 2x\,\mathrm{d}x = 4.$$

(Exercise 4) *Consider the graph surface* $S = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ z = g(x,y),\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in R\}$, *where* $R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ x^2 + y^2 < 1\}$ *and* $g(x,y) = x^2 - y^2$. *Define on* S *the standard upward-pointing orientation* $\hat{\boldsymbol{n}}$ *such that* $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} > 0$. *The boundary* ∂S *of* S *is a path, find a curve* $\vec{\mathbf{a}}(t)$ *parametrising it. Compute the path orientation* $\hat{\boldsymbol{\tau}}$ *of the parametrisation you found. State whether* $\hat{\boldsymbol{\tau}}$ *and the orientation of* ∂S *induced by the surface orientation* $\hat{\boldsymbol{n}}$ *of* S *coincide or not.*

A parametrisation of the boundary ∂R of the flat region R defining S is $\vec{\mathbf{b}}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}$. This is the usual circumference parametrisation (see e.g. pages 9–10 in the notes).

The points on S and on its boundary satisfy $z = g(x,y)$. In particular, the boundary can be written as

$$\partial S = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ z = g(x,y),\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \partial R\big\}.$$

So a possible parametrisation of ∂S is

$$\vec{\mathbf{a}}(t) = b_1(t)\hat{\boldsymbol{\imath}} + b_2(t)\hat{\boldsymbol{\jmath}} + g\big(b_1(t), b_2(t)\big)\hat{\boldsymbol{k}} = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + g(\cos t, \sin t)\hat{\boldsymbol{k}} = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + (\cos^2 t - \sin^2 t)\hat{\boldsymbol{k}},$$

for e.g. $t \in [0, 2\pi)$. (Note that this is the same trick we used in the parametrisation of the graph surface boundary in the proof of Stokes' theorem.) The orientation of $\vec{\mathbf{a}}$ is

$$\hat{\boldsymbol{\tau}}(\tau) = \frac{\frac{d\vec{\mathbf{a}}}{dt}(t)}{\left|\frac{d\vec{\mathbf{a}}}{dt}(t)\right|} = \frac{-\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} - 4\cos t \sin t\hat{\boldsymbol{k}}}{\sqrt{\sin^2 t + \cos^2 t + 16\sin^2 t\cos^2 t}}.$$

We have to check if this parametrisation has the orientation induced by $\hat{\boldsymbol{n}}$. We can draw S as in Figure 1 (the actual shape of S does not really matter). The induced parametrisation has to move anti-clockwise around S: according to its definition (see page 51 of the notes), moving along ∂S in the direction $\hat{\boldsymbol{\tau}}$, staying on the upper side of S (where $\hat{\boldsymbol{n}}$ lies), we must find S at our left. In particular, the induced orientation must be equal to $\hat{\boldsymbol{\jmath}}$ in $x = 1, y = 0$ (and $z = 1$).

The curve $\vec{\mathbf{a}}$ defined above passes through $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}$ for $t = 0$, and its orientation gives

$$\hat{\boldsymbol{\tau}}(0) = \frac{-\sin 0\,\hat{\boldsymbol{\imath}} + \cos 0\,\hat{\boldsymbol{\jmath}} - 4\cos 0\sin 0\hat{\boldsymbol{k}}}{\sqrt{\sin^2 0 + \cos^2 0 + 16\sin^2 0\cos^2 0}} = \hat{\boldsymbol{\jmath}}.$$

So the orientation $\hat{\boldsymbol{\tau}}$ of $\vec{\mathbf{a}}$ coincide with that induced by $(S, \hat{\boldsymbol{n}})$. (Of course, many other parametrisations are possible.)

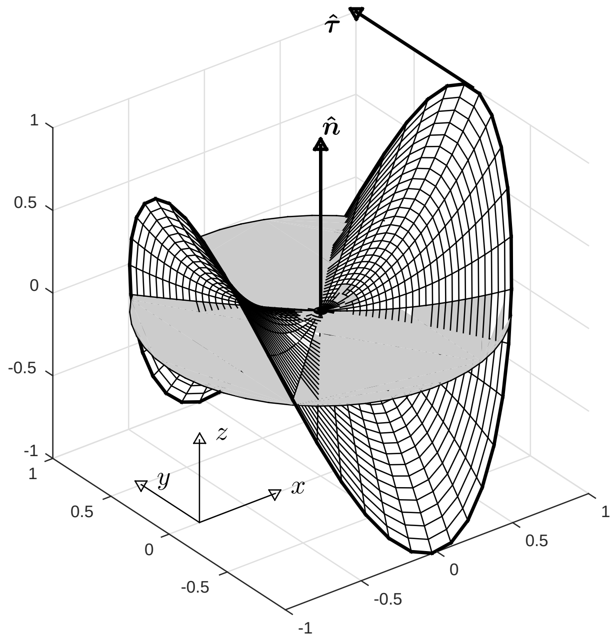

Figure 1: The surface S of Exercise (5). The unit normal vector $\hat{\boldsymbol{n}}$ denotes the (surface) orientation of S; the unit tangent vector $\hat{\boldsymbol{\tau}}$ denotes the (path) orientation of ∂S induced by $(S, \hat{\boldsymbol{n}})$: moving along ∂S in the direction of $\hat{\boldsymbol{\tau}}$, staying on the same side of $\hat{\boldsymbol{n}}$ (i.e. above) we see S at our left (see page 51). The grey disc is the region R.