Vector calculus MA3VC 2015–16: Assignment 2

MA3VC: Part 3 students only.
Handed out: Tuesday 17th November. Due: **Thursday 26th November, 12 noon**.
You can use formulas and identities from the lecture notes. Do not use red pen nor pencil.
Marking will be anonymous, so please write your name only on the "assessed work coversheet" and not on your work. Write your student number both on the back of the coversheet and each page of your work.
Total marks: 30. (10% of the total marks for the module.)

(Exercise 1 — 6 marks) Consider the square $Q = (0,1)^2 = \{x\hat{\boldsymbol\imath} + y\hat{\boldsymbol\jmath} \in \mathbb{R}^2,\ 0 < x < 1, 0 < y < 1\}$ and the change of variables

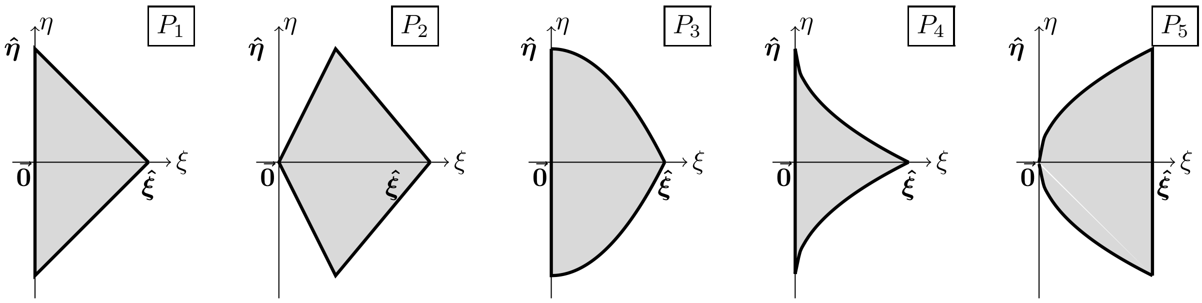
$$\vec{\mathbf{T}}(x,y) = \xi(x,y)\hat{\boldsymbol\xi} + \eta(x,y)\hat{\boldsymbol\eta} \qquad \text{where} \qquad \xi(x,y) = xy, \qquad \eta(x,y) = y^2 - x^2.$$

1. Compute the area of the transformed region $\vec{\mathbf{T}}(Q)$.

   Hint: Recall example 2.28 in the notes.

2. Which of the following regions corresponds to $\vec{\mathbf{T}}(Q)$? Justify your answer.

   Hint: the equations of the sides of $\vec{\mathbf{T}}(Q)$, obtained from those of the four sides of Q, may help.



(Exercise 2 — 14 marks) Let us fix the vector field $\vec{\mathbf{F}} = x(\hat{\boldsymbol\imath} + \hat{\boldsymbol k}) + 2y\hat{\boldsymbol\jmath}$.

1. Compute the line integral of $\vec{\mathbf{F}}$ on the straight segment Γ_S from $\hat{\boldsymbol\imath}$ to $\hat{\boldsymbol\jmath}$.

   Hint: recall Remark 1.24 on the parametrisation of paths.

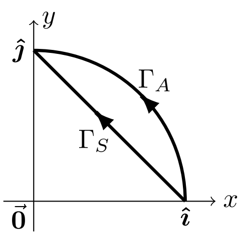
2. Compute the line integral of $\vec{\mathbf{F}}$ on the arc Γ_A of the unit circle $\{x^2 + y^2 = 1,\ z = 0\}$ from $\hat{\boldsymbol\imath}$ to $\hat{\boldsymbol\jmath}$.

3. Prove that, for all paths Γ running from $\hat{\boldsymbol\imath}$ to $\hat{\boldsymbol\jmath}$ and lying in the xy-plane $\{z = 0\}$, the equality $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ holds (where Γ_S is the segment from part 1 of the question).

   Hint: what is special in the parametrisation of a path lying in the xy-plane?

4. Find a path Γ_V from $\hat{\boldsymbol\imath}$ to $\hat{\boldsymbol\jmath}$ such that $\int_{\Gamma_V} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \neq \int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$.

   Hint: don't forget the statement shown in question 3 (even if you did not manage to prove it). Look for a simple path, you should be able to find one whose parametrisation's components are polynomials of degree at most two.



Turn the page for Exercise 3.

(Exercise 3 — 10 marks) Say which of the following statements are true.
Justify your answer.
(In case the statement is true, prove it, otherwise find a simple counterexample.)

1. Let the path Γ be part of the graph of a function $y = g(x)$ and $\vec{\mathbf{F}}$ be a conservative field. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.
2. Let Γ be a circle and $\vec{\mathbf{F}}$ an irrotational field defined in all of $\mathbb{R}^3$. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.
3. Let $\vec{\mathbf{F}}$ be a vector field perpendicular to the path Γ at each point. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.
4. Let $\vec{\mathbf{F}}$ be a vector field perpendicular to a surface S at each point. Then $\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = 0$.
5. Let $\vec{\mathbf{X}}$ be a chart of a parametric surface S. Then $\iint_S \frac{\partial \vec{\mathbf{X}}}{\partial u} \cdot \mathrm{d}\vec{\mathbf{S}} = 0$.