

# Vector calculus MA3VC 2016–17: Assignment 1

## SOLUTIONS AND FEEDBACK

**(Exercise 1)** Prove that if  $f$  is a (smooth) scalar field and  $\vec{\mathbf{G}}$  is an irrotational vector field, then

$$(\vec{\nabla}f \times \vec{\mathbf{G}})f$$

is solenoidal.

**Hint:** do not expand in coordinates and partial derivatives. Use instead the vector differential identities of §1.4 and the properties of the vector product from §1.1.2 (recall in particular Exercise 1.15).

(Version 1) We compute the divergence of the vector field  $(\vec{\nabla}f \times \vec{\mathbf{G}})f$  using the product rules (29) and (30) for the divergence:

$$\begin{aligned} \vec{\nabla} \cdot \left( (\vec{\nabla}f \times \vec{\mathbf{G}})f \right) &\stackrel{(29)}{=} \vec{\nabla}f \cdot (\vec{\nabla}f \times \vec{\mathbf{G}}) + f\vec{\nabla} \cdot (\vec{\nabla}f \times \vec{\mathbf{G}}) \\ &\stackrel{(30)}{=} \vec{\nabla}f \cdot (\vec{\nabla}f \times \vec{\mathbf{G}}) + f(\vec{\nabla} \times \vec{\nabla}f) \cdot \vec{\mathbf{G}} - f\vec{\nabla}f \cdot (\vec{\nabla} \times \vec{\mathbf{G}}). \end{aligned}$$

The first term in this expression vanishes<sup>1</sup> because of the identities  $\vec{\mathbf{u}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{w}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{u}})$  (by Exercise 1.15) and  $\vec{\mathbf{u}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$  (by the anticommutativity of the vector product (3)), which hold for all  $\vec{\mathbf{u}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ . Applying these formulas with  $\vec{\mathbf{u}} = \vec{\nabla}f$  and  $\vec{\mathbf{w}} = \vec{\mathbf{G}}$  we get  $\vec{\nabla}f \cdot (\vec{\nabla}f \times \vec{\mathbf{G}}) = \vec{\mathbf{G}} \cdot (\vec{\nabla}f \times \vec{\nabla}f) = \vec{\mathbf{G}} \cdot \vec{\mathbf{0}} = 0$ .

The second term vanishes because of the “curl-grad identity” (26):  $\vec{\nabla} \times \vec{\nabla}f = \vec{\mathbf{0}}$ .

The last term vanishes because  $\vec{\mathbf{G}}$  is irrotational:  $\vec{\nabla} \times \vec{\mathbf{G}} = \vec{\mathbf{0}}$ .

So we conclude that the field  $(\vec{\nabla}f \times \vec{\mathbf{G}})f$  is solenoidal because its divergence vanishes:

$$\vec{\nabla} \cdot \left( (\vec{\nabla}f \times \vec{\mathbf{G}})f \right) = \underbrace{\vec{\mathbf{G}} \cdot (\vec{\nabla}f \times \vec{\nabla}f)}_{=\vec{\mathbf{0}}} + f \underbrace{(\vec{\nabla} \times \vec{\nabla}f) \cdot \vec{\mathbf{G}}}_{=\vec{\mathbf{0}}} - f \underbrace{\vec{\nabla}f \cdot (\vec{\nabla} \times \vec{\mathbf{G}})}_{=\vec{\mathbf{0}}} = 0.$$

(Version 2) Alternatively, one could expand the field in coordinates. This solution is of course much more complicated and prone to errors than the previous one.

We first write the product rule for products of three scalar fields, which is proved by applying twice the usual product rule for partial derivatives (8):

$$\frac{\partial}{\partial x}(fgh) = \frac{\partial}{\partial x}((fg)h) \stackrel{(8)}{=} \frac{\partial(fg)}{\partial x}h + fg\frac{\partial h}{\partial x} \stackrel{(8)}{=} \left( \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x} \right)h + fg\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x}gh + f\frac{\partial g}{\partial x}h + fg\frac{\partial h}{\partial x}. \quad (\spadesuit)$$

Combining this formula with the definitions of gradient (10), vector product (2), divergence (22) and curl (23), using Clairaut’s theorem (17) (i.e.  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ ) and rearranging the terms, we obtain:

$$\begin{aligned} \vec{\nabla} \cdot \left( (\vec{\nabla}f \times \vec{\mathbf{G}})f \right) &\stackrel{(10)}{=} \vec{\nabla} \cdot \left( \left( \left( \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \right) \times \vec{\mathbf{G}} \right) f \right) \\ &\stackrel{(2)}{=} \vec{\nabla} \cdot \left( \left( \left( \frac{\partial f}{\partial y}G_3 - \frac{\partial f}{\partial z}G_2 \right)\hat{\mathbf{i}} + \left( \frac{\partial f}{\partial z}G_1 - \frac{\partial f}{\partial x}G_3 \right)\hat{\mathbf{j}} + \left( \frac{\partial f}{\partial x}G_2 - \frac{\partial f}{\partial y}G_1 \right)\hat{\mathbf{k}} \right) f \right) \\ &= \vec{\nabla} \cdot \left( \left( f\frac{\partial f}{\partial y}G_3 - f\frac{\partial f}{\partial z}G_2 \right)\hat{\mathbf{i}} + \left( f\frac{\partial f}{\partial z}G_1 - f\frac{\partial f}{\partial x}G_3 \right)\hat{\mathbf{j}} + \left( f\frac{\partial f}{\partial x}G_2 - f\frac{\partial f}{\partial y}G_1 \right)\hat{\mathbf{k}} \right) \\ &\stackrel{(22)}{=} \frac{\partial}{\partial x} \left( f\frac{\partial f}{\partial y}G_3 - f\frac{\partial f}{\partial z}G_2 \right) + \frac{\partial}{\partial y} \left( f\frac{\partial f}{\partial z}G_1 - f\frac{\partial f}{\partial x}G_3 \right) + \frac{\partial}{\partial z} \left( f\frac{\partial f}{\partial x}G_2 - f\frac{\partial f}{\partial y}G_1 \right) \\ &\stackrel{(\spadesuit)}{=} \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} G_3 + f \frac{\partial^2 f}{\partial x \partial y} G_3 + f \frac{\partial f}{\partial y} \frac{\partial G_3}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} G_2 - f \frac{\partial^2 f}{\partial x \partial z} G_2 - f \frac{\partial f}{\partial z} \frac{\partial G_2}{\partial x} \\ &\quad + \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} G_1 + f \frac{\partial^2 f}{\partial y \partial z} G_1 + f \frac{\partial f}{\partial z} \frac{\partial G_1}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} G_3 - f \frac{\partial^2 f}{\partial y \partial x} G_3 - f \frac{\partial f}{\partial x} \frac{\partial G_3}{\partial y} \\ &\quad + \frac{\partial f}{\partial z} \frac{\partial f}{\partial x} G_2 + f \frac{\partial^2 f}{\partial z \partial x} G_2 + f \frac{\partial f}{\partial x} \frac{\partial G_2}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial f}{\partial y} G_1 - f \frac{\partial^2 f}{\partial z \partial y} G_1 - f \frac{\partial f}{\partial y} \frac{\partial G_1}{\partial z} \\ &= G_1 \underbrace{\left( \frac{\partial f}{\partial y} \frac{\partial f}{\partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial f}{\partial z} \frac{\partial f}{\partial y} - f \frac{\partial^2 f}{\partial z \partial y} \right)}_{=0, (17)} \\ &\quad + G_2 \underbrace{\left( -\frac{\partial f}{\partial x} \frac{\partial f}{\partial z} - f \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial f}{\partial z} \frac{\partial f}{\partial x} + f \frac{\partial^2 f}{\partial z \partial x} \right)}_{=0, (17)} \end{aligned}$$

<sup>1</sup>Equivalently  $\vec{\nabla}f \cdot (\vec{\nabla}f \times \vec{\mathbf{G}}) = 0$  because the vectors  $\vec{\nabla}f$ ,  $\vec{\mathbf{G}}$ ,  $\vec{\nabla}f \times \vec{\mathbf{G}}$  are linearly dependent, or because  $\vec{\nabla}f \times \vec{\mathbf{G}}$  is orthogonal to  $\vec{\nabla}f$ .

$$\begin{aligned}
& + G_2 \underbrace{\left( \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} + f \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial f}{\partial y} \frac{\partial f}{\partial x} - f \frac{\partial^2 f}{\partial y \partial x} \right)}_{=0, (17)} \\
& + f \frac{\partial f}{\partial x} \left( \frac{\partial G_2}{\partial z} - \frac{\partial G_3}{\partial y} \right) + f \frac{\partial f}{\partial y} \left( \frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z} \right) + f \frac{\partial f}{\partial z} \left( \frac{\partial G_1}{\partial y} - \frac{\partial G_2}{\partial x} \right) \\
& \stackrel{(23)}{=} f \frac{\partial f}{\partial x} (-\vec{\nabla} \times \vec{\mathbf{G}})_1 + f \frac{\partial f}{\partial y} (-\vec{\nabla} \times \vec{\mathbf{G}})_2 + f \frac{\partial f}{\partial z} (-\vec{\nabla} \times \vec{\mathbf{G}})_3 \\
& = 0 \quad \text{since } \vec{\nabla} \times \vec{\mathbf{G}} = \vec{\mathbf{0}}.
\end{aligned}$$

**(Exercise 2)** Demonstrate that the field  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  is indeed solenoidal for  $f = xyz$  and  $\vec{\mathbf{G}} = \vec{\mathbf{r}}$ .

We compute all the terms in the expression  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  and its divergence:

$$\begin{aligned}
\vec{\mathbf{G}} &= \vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}, & f &= xyz, \\
\vec{\nabla} f &= yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}}, \\
\vec{\nabla} f \times \vec{\mathbf{G}} &= (xz^2 - xy^2)\hat{\mathbf{i}} + (yx^2 - yz^2)\hat{\mathbf{j}} + (zy^2 - zx^2)\hat{\mathbf{k}}, \\
(\vec{\nabla} f \times \vec{\mathbf{G}})f &= (x^2yz^3 - x^2y^3z)\hat{\mathbf{i}} + (x^3y^2z - xy^2z^3)\hat{\mathbf{j}} + (xy^3z^2 - x^3yz^2)\hat{\mathbf{k}}, \\
\vec{\nabla} \cdot ((\vec{\nabla} f \times \vec{\mathbf{G}})f) &= 2xyz^3 - 2xy^3z + 2x^3yz - 2xy^3z + 2xy^3z - 2x^3yz = 0.
\end{aligned}$$

**(Exercise 3)** Consider the vector field  $\vec{\mathbf{F}} = x^2y(\hat{\mathbf{i}} - \hat{\mathbf{k}}) - xy^2\hat{\mathbf{j}}$ .

1. Show that  $\vec{\mathbf{F}}$  is not conservative.

*Hint:* Use the relation between conservative and irrotational fields.

2. Find a vector potential  $\vec{\mathbf{A}}$  for  $\vec{\mathbf{F}}$ .

*Hint:* Look for a vector potential that depends on two coordinate variables only.

3. Let  $\vec{\mathbf{B}} \neq \vec{\mathbf{A}}$  be any other vector potential of  $\vec{\mathbf{F}}$ . Prove that the difference  $\vec{\mathbf{A}} - \vec{\mathbf{B}}$  is irrotational.

4. Find a vector field  $\vec{\mathbf{B}}$  that is simultaneously (i) a vector potential for  $\vec{\mathbf{F}}$  and (ii) solenoidal.

*Hint:* Given  $\vec{\mathbf{A}}$  from the previous question, find a scalar field  $\alpha$  such that  $\vec{\mathbf{B}} = \vec{\mathbf{A}} + \vec{\nabla}\alpha$  satisfies the request.

5. Can you find a field  $\vec{\mathbf{G}}$  that admits  $\vec{\mathbf{F}}$  as vector potential? How many such fields exist?

1. Since  $\vec{\nabla} \times \vec{\mathbf{F}} = -x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + (-y^2 - x^2)\hat{\mathbf{k}} \neq \vec{\mathbf{0}}$ ,  $\vec{\mathbf{F}}$  is not irrotational. By the implication on page 26 of the notes (conservative  $\Rightarrow$  irrotational),  $\vec{\mathbf{F}}$  is not conservative.

2. The vector potential  $\vec{\mathbf{A}}$  has to satisfy  $\vec{\nabla} \times \vec{\mathbf{A}} = \vec{\mathbf{F}}$ , which, by definition (23) of the curl, is equivalent to

$$\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = F_1 = x^2y, \quad \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = F_2 = -xy^2, \quad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = F_3 = -x^2y.$$

Since the right-hand sides do not depend on  $z$ , we try to find  $\vec{\mathbf{A}}$  independent of  $z$ , so  $\frac{\partial A_2}{\partial z} = \frac{\partial A_1}{\partial z} = 0$  and

$$\frac{\partial A_3}{\partial y} = x^2y, \quad -\frac{\partial A_3}{\partial x} = -xy^2, \quad \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = -x^2y.$$

From the first two equations we have  $A_3 = \frac{1}{2}x^2y^2 + \lambda$ , where  $\lambda$  is a constant. From the last equation  $\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = -x^2y$  we see that we can choose (only) one of  $A_1$  and  $A_2$  to be zero. Choosing  $A_2 = 0$  we get  $A_1 = \frac{1}{2}x^2y^2 + \mu$ , where  $\mu$  is another constant. Choosing  $\lambda = \mu = 0$  (any other value would be OK), we obtain  $\vec{\mathbf{A}} = \frac{1}{2}x^2y^2(\hat{\mathbf{i}} + \hat{\mathbf{k}})$ .

If instead we chose  $A_1 = 0$  we would have obtained  $\vec{\mathbf{A}} = -\frac{1}{3}x^3y\hat{\mathbf{j}} + \frac{1}{2}x^2y^2\hat{\mathbf{k}}$ . Of course, many other potentials are acceptable.

3.  $\vec{\nabla} \times (\vec{\mathbf{A}} - \vec{\mathbf{B}}) = \vec{\nabla} \times \vec{\mathbf{A}} - \vec{\nabla} \times \vec{\mathbf{B}} = \vec{\mathbf{F}} - \vec{\mathbf{F}} = \vec{\mathbf{0}}$ , which means that the difference is irrotational.

4. From the hint, we look for some  $\alpha$  such that  $\vec{\mathbf{B}} = \vec{\mathbf{A}} + \vec{\nabla}\alpha$  is as desired. We want  $\vec{\mathbf{B}}$  to be solenoidal and a vector potential for  $\vec{\mathbf{F}}$ , which means

$$0 = \vec{\nabla} \cdot \vec{\mathbf{B}} = \vec{\nabla} \cdot (\vec{\mathbf{A}} + \vec{\nabla}\alpha) = \vec{\nabla} \cdot \vec{\mathbf{A}} + \vec{\nabla} \cdot (\vec{\nabla}\alpha), \quad \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{B}} = \vec{\nabla} \times (\vec{\mathbf{A}} + \vec{\nabla}\alpha) = \vec{\nabla} \times \vec{\mathbf{A}} + \vec{\nabla} \times (\vec{\nabla}\alpha).$$

Since by (26)  $\vec{\nabla} \times (\vec{\nabla}\alpha) = \vec{\mathbf{0}}$  and  $\vec{\nabla}\vec{\mathbf{A}} = \vec{\mathbf{F}}$ , the second equation is always satisfied. Using identity (24)  $\vec{\nabla} \cdot (\vec{\nabla}\alpha) = \Delta\alpha$ , the first equation can be rewritten as  $\Delta\alpha = -\vec{\nabla} \cdot \vec{\mathbf{A}}$ .

If in the previous step we found  $\vec{\mathbf{A}} = \frac{1}{2}x^2y^2(\hat{\mathbf{i}} + \hat{\mathbf{k}})$ , then  $\Delta\alpha = -\vec{\nabla} \cdot \vec{\mathbf{A}} = -xy^2$ , so we can choose  $\alpha = -\frac{1}{12}xy^4$  (check:  $\Delta\alpha = \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} = 0 - xy^2$ ). From this we have

$$\vec{\mathbf{B}} = \frac{1}{2}x^2y^2(\hat{\mathbf{i}} + \hat{\mathbf{k}}) + \vec{\nabla}\left(-\frac{1}{12}xy^4\right) = \frac{1}{2}x^2y^2(\hat{\mathbf{i}} + \hat{\mathbf{k}}) - \frac{1}{12}y^4\hat{\mathbf{i}} - \frac{1}{3}xy^3\hat{\mathbf{j}} = \boxed{\left(\frac{1}{2}x^2y^2 - \frac{1}{12}y^4\right)\hat{\mathbf{i}} - \frac{1}{3}xy^3\hat{\mathbf{j}} + \frac{1}{2}x^2y^2\hat{\mathbf{k}}}.$$

Then it is easy to double check that  $\vec{\nabla} \cdot \vec{\mathbf{B}} = 0$  and  $\vec{\nabla} \times \vec{\mathbf{B}} = \vec{\mathbf{F}}$ .

If instead in the previous subtask we computed  $\vec{\mathbf{A}} = -\frac{1}{3}x^3y\hat{\mathbf{j}} + \frac{1}{2}x^2y^2\hat{\mathbf{k}}$ , then  $\Delta\alpha = -\vec{\nabla} \cdot \vec{\mathbf{A}} = \frac{1}{3}x^3$  so we can choose  $\alpha = \frac{1}{60}x^5$  and  $\boxed{\vec{\mathbf{B}} = \frac{1}{12}x^4\hat{\mathbf{i}} - \frac{1}{3}x^3y\hat{\mathbf{j}} + \frac{1}{2}x^2y^2\hat{\mathbf{k}}}$ .

5. If  $\vec{\mathbf{G}}$  admits  $\vec{\mathbf{F}}$  as vector potential then  $\boxed{\vec{\mathbf{G}} = \vec{\nabla} \times \vec{\mathbf{F}} = -x^2\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}} + (-y^2 - x^2)\hat{\mathbf{k}}}$ . Thus there are no other fields admitting  $\vec{\mathbf{F}}$  as vector potential (while there are many other vector potentials for  $\vec{\mathbf{G}}$ ).

### MA3VC: Feedback after grading assignment 1

Check carefully the points below and all the corrections in your assignment; even if you got full marks, your solution (and its presentation) can probably be improved. See also page 108 in the notes.

In general, in your coursework, a red check mark  $\checkmark$  means “correct”, a  $\times$  mark means “error”, a check mark in brackets ( $\checkmark$ ) means “correct step leading to a wrong solution due to previous errors”.

If you have any question or comment about the assignment, the solutions or the marking, please do ask me.

#### Exercise 1:

Here I have found many serious mistakes. If you did not get full marks, please revise carefully the solutions.

- The most common mistake was in the use of the differential operators and the **nabla symbol** “ $\vec{\nabla}$ ”.

**The nabla symbol  $\vec{\nabla}$  cannot be treated as a vector!** It is just the symbol we use to write gradients  $\vec{\nabla}f$ , divergences  $\vec{\nabla} \cdot \vec{\mathbf{F}}$  and curls  $\vec{\nabla} \times \vec{\mathbf{F}}$ . The gradient  $\vec{\nabla}f$  of a scalar field is a vector field, while  $\vec{\nabla}$  alone is not. This is similar to the fact that “ $\cos\pi$ ” is a number, but “ $\cos$ ” alone is not.

Many frequent errors are consequences of this.

- $\vec{\nabla}$  is not “multiplied” but “applied”:  $\vec{\nabla} \cdot \vec{\mathbf{F}}$  and  $\vec{\nabla} \times \vec{\mathbf{F}}$  are divergence and curl of a field  $\vec{\mathbf{F}}$ . If you call them “dot product” and “cross product” you do a mistake.
- The nabla symbol can appear in a formula *only* as part of a gradient, a divergence or a curl. Otherwise its use is wrong. For example, terms like  $(\vec{\nabla} \times \vec{\mathbf{F}})$  and  $(\vec{\nabla}f)$  are correct, while  $(\vec{\mathbf{F}} \times \vec{\nabla})$  and  $(f\vec{\nabla})$  are meaningless (exactly as  $\sin\pi$  is 0, while  $\pi\sin$  is meaningless).
- Identities and properties that are true for vectors are not necessarily true for  $\vec{\nabla}$ . For example,  $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$  holds for any three vectors  $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^3$ . But in general  $\vec{\nabla} \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}}) \neq \vec{\mathbf{G}} \cdot (\vec{\nabla} \times \vec{\mathbf{F}})$ . The expression  $\vec{\nabla} \cdot (\vec{\mathbf{F}} \times \vec{\mathbf{G}})$  is the divergence of the vector product between two fields  $\vec{\mathbf{F}}$  and  $\vec{\mathbf{G}}$ , to compute it you need to use the appropriate product rule (i.e. equation (30)). Many people committed this mistake.
- You cannot change the position of the symbol  $\vec{\nabla}$  in a formula using some sort of “commutativity”. For example, the products  $f\vec{\nabla} \cdot \vec{\mathbf{G}}$  and  $\vec{\nabla}f \cdot \vec{\mathbf{G}}$  are different: the former contains some partial derivatives of  $\vec{\mathbf{G}}$  (its divergence), while the latter contains the partial derivatives of  $f$  (its gradient). Thus you cannot equate them using some commutativity of products, as  $\vec{\nabla}$  is not “multiplied” but “applied”.

Tip: the use of the notation  $\vec{grad}f$ ,  $div\vec{\mathbf{F}}$  and  $\vec{curl}\vec{\mathbf{F}}$  instead of  $\vec{\nabla}f$ ,  $\vec{\nabla} \cdot \vec{\mathbf{F}}$  and  $\vec{\nabla} \times \vec{\mathbf{F}}$  can prevent many of these errors. E.g. the two terms in the last example would read  $f\,div\vec{\mathbf{G}}$  and  $\vec{\mathbf{G}} \cdot \vec{grad}f$ , which are evidently different.

- Many students computed  $[\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{\mathbf{G}})]f$  instead of  $\vec{\nabla} \cdot [(\vec{\nabla} f \times \vec{\mathbf{G}})f]$ . In order to prove that the field  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  is solenoidal,  $f$  cannot be moved out of the divergence freely, the product rule (29) for the divergence has to be used.
- Many students used some sort of distributivity property in a completely wrong way, rewriting  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  as  $\vec{\nabla} f f \times \vec{\mathbf{G}} f$ . The distributivity property relates addition and multiplication ( $(a+b)c = ac + bc$  for  $a, b, c \in \mathbb{R}$ ), while in  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  only products are present (a vector product and a vector-times-scalar product). This would be the same as saying that  $(ab)c = acb$ .
- Writing  $\vec{\nabla} f f$  is slightly ambiguous. All students who wrote this, treated it first as  $(\vec{\nabla} f)f$  and then as  $\vec{\nabla}(ff)$ , which are not the same. Use brackets to avoid mistakes!
- Some students tried to solve exercise 1 by expanding the field in coordinates (version 2 above). Nobody got close to the solution and all did mistakes in the use of partial derivatives. The symbol  $\frac{\partial}{\partial x}$  can not be moved around in an expression freely and fields cannot be moved in and out the derivative. For example,  $\frac{\partial f}{\partial y} G_3$  and  $\frac{\partial(fG_3)}{\partial y}$  are not the same: the former is the product of two scalar fields (the partial derivative of  $f$  in  $y$  and the third component of  $\vec{\mathbf{G}}$ ), while the latter is the partial derivative of the product of  $f$  and  $G_3$ . On the other hand, writing things like  $\frac{\partial f G_3}{\partial y}$  is ambiguous, thus incorrect, and led many to errors.
- In exercises where you have to “prove” something, write down explicitly which tool you use at each step (e.g. product rule, this or that identity. . .).
- Some students wrote that “ $\vec{\nabla} f = \vec{\mathbf{0}}$  because  $f$  is smooth”. This is not correct, smooth fields are not necessarily constant.

### Exercise 2:

- As in the first exercise, many students computed  $[\vec{\nabla} \cdot (\vec{\nabla} f \times \vec{\mathbf{G}})]f$  instead of  $\vec{\nabla} \cdot [(\vec{\nabla} f \times \vec{\mathbf{G}})f]$ . Some others computed the scalar field  $(\vec{\nabla} f \times \vec{\mathbf{G}}) \cdot \vec{\mathbf{G}}$  (which is zero) instead of the correct vector field. Check carefully if what you are computing is what you are asked to.
- When you are asked to **demonstrate** an identity for a certain choice of fields, you have to compute both its left-hand side and its right-hand side, and verify that the two results coincide. This exercise was even easier, as the right-hand side is zero. The best strategy is simply to compute and list all the terms appearing in the final identity. In this exercise one can write four steps: (i)  $\vec{\nabla} f$ , (ii)  $\vec{\nabla} f \times \vec{\mathbf{G}}$ , (iii)  $(\vec{\nabla} f \times \vec{\mathbf{G}})f$  and (iv)  $\vec{\nabla} \cdot ((\vec{\nabla} f \times \vec{\mathbf{G}})f)$ . You cannot use the identity itself to deduce that the left-hand side is zero (or prove again the identity), this is a circular argument.
- The position vector is  $\vec{\mathbf{r}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ ; its components are the independent variables  $x$ ,  $y$  and  $z$ . You will not be able to solve this exercise and many others if you ignore this.

### Exercise 3:

In the first part there is no need to involve the shape of the domain of definition of  $\vec{\mathbf{F}}$ . The implication “conservative $\Rightarrow$ irrotational” (thus also “not irrotational $\Rightarrow$ not conservative”) holds for any vector field. The implication “irrotational $\Rightarrow$ conservative” requires further assumptions but was not needed here.