Vector calculus MA3VC 2016–17: Assignment 3
SOLUTIONS AND FEEDBACK

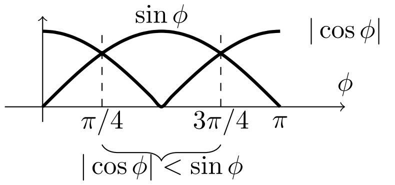
(Exercise 1) *Compute the volume of the portion D of the unit ball $B = \{\vec{\mathbf{r}} \in \mathbb{R}^3; |\vec{\mathbf{r}}| < 1\}$ that is not contained in the double cone $C = \{\vec{\mathbf{r}} \in \mathbb{R}^3; z^2 \geq x^2 + y^2\}$.*

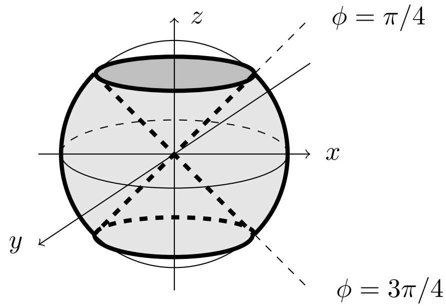
Hint: *Use a set of special coordinates.*

We simply compute the volume as the triple integral of 1 over D. We first note that the points of D are those with magnitude smaller than 1 and colatitude between $\pi/4$ and $3\pi/4$:

$$\begin{aligned} D &= \{|\vec{\mathbf{r}}| < 1, z^2 < x^2 + y^2\} \\ &= \{\rho < 1, \rho^2 \cos^2\phi < \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta\} \qquad \text{from definition of spherical coordinates (87)} \\ &= \{\rho < 1, |\cos\phi| < |\sin\phi|\} \\ &= \{\rho < 1, \pi/4 < \phi < 3\pi/4\}, \end{aligned}$$

$$\mathrm{Vol}(D) = \iiint_D 1\,\mathrm{d}V = \int_0^1 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{-\pi}^{\pi} \rho^2 \sin\phi \,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\rho = \int_0^1 \rho^2\,\mathrm{d}\rho \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\phi\,\mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\theta = \boxed{\frac{1}{3}\sqrt{2}(2\pi)} \approx 2.96.$$





(Exercise 2) *Fix a positive number k. Consider the so called "time-harmonic Maxwell equation" for a vector field $\vec{\mathbf{F}}$*

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - k^2 \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

and the "Helmholtz equation" for a scalar field f

$$\Delta f + k^2 f = 0.$$

Prove that a smooth vector field $\vec{\mathbf{F}}$ satisfies Maxwell's equation if and only if *it is solenoidal and its three components satisfy the Helmholtz equation.*

Hint: *Recall the identities proved in Section 1.4 of the notes.*

We first note that by identity (27)

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{F}}) - \vec{\boldsymbol{\Delta}}\vec{\mathbf{F}}.$$

We prove the implication "*Helmholtz + solenoidal ⇒ Maxwell*". Assume that the three components F_1, F_2 and F_3 of $\vec{\mathbf{F}}$ satisfy the Helmholtz equation and that $\vec{\mathbf{F}}$ is solenoidal, i.e. $\vec{\nabla} \cdot \vec{\mathbf{F}} = 0$. Then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - k^2\vec{\mathbf{F}} \overset{(27)}{=} \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{\mathbf{F}}}_{=0}) - \vec{\boldsymbol{\Delta}}\vec{\mathbf{F}} - k^2\vec{\mathbf{F}} = (\underbrace{-\Delta F_1 - k^2 F_1}_{=0})\hat{\boldsymbol{\imath}} + (\underbrace{-\Delta F_2 - k^2 F_2}_{=0})\hat{\boldsymbol{\jmath}} + (\underbrace{-\Delta F_3 - k^2 F_3}_{=0})\hat{\boldsymbol{k}} = \vec{\mathbf{0}},$$

which means that $\vec{\mathbf{F}}$ satisfies Maxwell's equation.

Now we prove the second implication, namely "*Maxwell ⇒ Helmholtz + solenoidal*". Assume that $\vec{\mathbf{F}}$ satisfies the Maxwell equation, $\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - k^2\vec{\mathbf{F}} = 0$. Then $\vec{\mathbf{F}}$ is solenoidal by the div-curl identity

$$\vec{\nabla} \cdot \vec{\mathbf{F}} = \vec{\nabla} \cdot \Big(\frac{1}{k^2}\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})\Big) = \frac{1}{k^2}\vec{\nabla} \cdot \Big(\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})\Big) = 0.$$

By identity (27), $\vec{\mathbf{F}}$ satisfies

$$-\vec{\boldsymbol{\Delta}}\vec{\mathbf{F}} - k^2\vec{\mathbf{F}} = \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{\mathbf{F}}}_{=0}) - k^2\vec{\mathbf{F}} = \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - k^2\vec{\mathbf{F}} = \vec{\mathbf{0}}.$$

From the definition of the vector Laplacian (21), this equation can be expanded in components as:

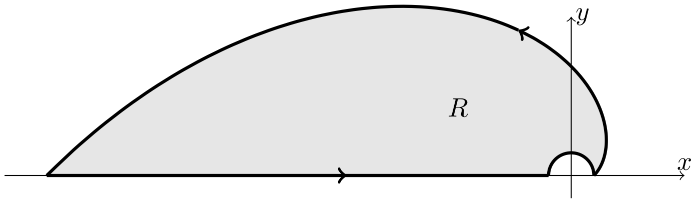
$$-\Delta F_1 - k^2 F_1 = 0, \qquad -\Delta F_2 - k^2 F_2 = 0, \qquad -\Delta F_3 - k^2 F_3 = 0,$$

which complete the assertion.

(Exercise 3) *Compute the line integral of* $\vec{\mathbf{G}} = -(x+y+x^3+y^2)\hat{\boldsymbol{\imath}} + (x+y-2xy)\hat{\boldsymbol{\jmath}}$ *along the boundary* ∂R *of the region* R *depicted in the figure below, which is defined in polar coordinates by*

$$R = \left\{\vec{\mathbf{r}} \in \mathbb{R}^2,\ 0 < \log r < \theta < \pi\right\}.$$

Assume that the path ∂R *is oriented in the anticlockwise direction.*
Hint: *Do not try to compute the integral by brute force; use instead some important theorem.*



The region can be written as $R = \{\vec{\mathbf{r}} \in \mathbb{R}^2,\ 1 < r < \mathrm{e}^\theta,\ 0 < \theta < \pi\}$.
By Green's theorem

$$\begin{aligned}
\int_{\partial R} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \iint_R (\vec{\nabla} \times \vec{\mathbf{G}}) \cdot \hat{\boldsymbol{k}} \,\mathrm{d}A \\
&= \iint_R (2\hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{k}} \,\mathrm{d}A \\
&= \iint_R 2 \,\mathrm{d}A \qquad \text{(the integral is twice the area of } R) \\
&= \int_0^\pi \int_1^{\mathrm{e}^\theta} 2r \,\mathrm{d}r \,\mathrm{d}\theta = \int_0^\pi r^2\Big|_1^{\mathrm{e}^\theta} \mathrm{d}\theta = \int_0^\pi (\mathrm{e}^{2\theta} - 1)\,\mathrm{d}\theta = \frac{1}{2}\mathrm{e}^{2\theta} - \theta\Big|_0^\pi = \boxed{\frac{1}{2}\mathrm{e}^{2\pi} - \pi - \frac{1}{2}} \approx 264.104.
\end{aligned}$$

Alternatively, one can directly compute the line integral. We first decompose the field as $\vec{\mathbf{G}} = \vec{\mathbf{H}} + \vec{\mathbf{M}}$, for $\vec{\mathbf{H}} = -y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$ and $\vec{\mathbf{M}} = -(x+x^3+y^2)\hat{\boldsymbol{\imath}} + (y-2xy)\hat{\boldsymbol{\jmath}}$, where $\vec{\mathbf{M}}$ is conservative. Since ∂R is a loop, the line integral $\oint_{\partial R} \vec{\mathbf{M}} \cdot \mathrm{d}\hat{\boldsymbol{r}} = 0$, by the fundamental theorem of vector calculus (or Lemma 2.19).

As suggested by the figure, in order to compute the line integral of $\vec{\mathbf{H}}$ we split ∂R in three parts: the straight segment Γ_S, the half circle Γ_a, the upper curvilinear part Γ_b. These correspond to the three inequalities in the definition of R: Γ_S to $\theta < \pi$, Γ_a to $0 < \log r$ and Γ_b to $\log r < \theta$.

On the segment Γ_S from $-\mathrm{e}^\pi\hat{\boldsymbol{\imath}}$ to $-\hat{\boldsymbol{\imath}}$, the tangential component of $\vec{\mathbf{H}}$ (namely its first component) is zero, so the integral $\int_{\Gamma_S} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}}$ vanishes.

The half circle Γ_a and the upper part Γ_b of ∂R can be parametrised as follows:

$$\begin{aligned}
&\vec{\mathbf{a}}(t) = -\cos t\hat{\boldsymbol{\imath}} + \sin t\hat{\boldsymbol{\jmath}}, \qquad 0 < t < \pi, \\
&\iint_{\Gamma_a} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}} = \int_0^\pi (-\sin t\hat{\boldsymbol{\imath}} - \cos t\hat{\boldsymbol{\jmath}}) \cdot (\sin t\hat{\boldsymbol{\imath}} + \cos t\hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^\pi -1\,\mathrm{d}t = -\pi, \\
&\vec{\mathbf{b}}(t) = \mathrm{e}^t\cos t\hat{\boldsymbol{\imath}} + \mathrm{e}^t\sin t\hat{\boldsymbol{\jmath}}, \qquad 0 < t < \pi, \\
&\int_{\Gamma_b} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}} = \int_0^\pi (-\mathrm{e}^t\sin t\hat{\boldsymbol{\imath}} + \mathrm{e}^t\cos t\hat{\boldsymbol{\jmath}}) \cdot \Big(\mathrm{e}^t(\cos t - \sin t)\hat{\boldsymbol{\imath}} + \mathrm{e}^t(\cos t + \sin t)\hat{\boldsymbol{\jmath}}\Big)\,\mathrm{d}t = \int_0^\pi \mathrm{e}^{2t}\,\mathrm{d}t = \frac{1}{2}(\mathrm{e}^{2\pi} - 1).
\end{aligned}$$

(Note that the minus sign in the parametrisation of the half circle depends on the fact that we are integrating in the opposite direction than the usual one.) Putting everything together, we find the value already obtained by Green's theorem:

$$\int_{\partial R} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\partial R} \vec{\mathbf{H}} \cdot \mathrm{d}\vec{\mathbf{r}} + \underbrace{\int_{\partial R} \vec{\mathbf{M}} \cdot \mathrm{d}\vec{\mathbf{r}}}_{=0} = \int_{\Gamma_a} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}} + \int_{\Gamma_b} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}} + \underbrace{\int_{\Gamma_S} \vec{\mathbf{H}} \cdot \mathrm{d}\hat{\boldsymbol{r}}}_{=0} = \frac{1}{2}(\mathrm{e}^{2\pi} - 1) - \pi + 0.$$

MA3VC: Feedback after grading assignment 3

Check carefully the points below and all the corrections in your assignment; even if you got full marks, your solution can probably be improved. See also page 110 in the notes.

In your coursework, a red check mark ✓ means "correct", a × mark means "error", a check mark in brackets (✓) means "correct step leading to a wrong solution due to previous errors".

If you have any question or comment about the assignment, the solutions or the marking, please do ask me.

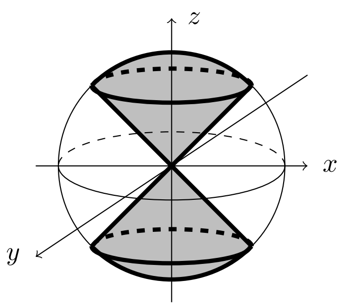
Exercise 1: The main difficulty (and source of errors) in this exercise was in the identification of the extremes of integration. The solution proposed above simply writes the domain in spherical coordinates as $D = \{\rho < 1,\ \pi/4 < \phi < 3\pi/4\}$ and then uses this expression to compute the integral. Many students computed the volume of the unit ball ($4\pi/3$) and then subtracted the volume of the double cone. However some did mistakes in the computation of the volume of the cone.

To proceed this way one has to compute the volume of the intersection $C \cap B$ of the double cone with the unit ball, and subtract this from the volume of the ball. (Note that the double cone $C = \{\vec{\mathbf{r}} \in \mathbb{R}^3; z^2 \geq x^2 + y^2\}$ is unbounded and has infinite volume.) However some students computed the volume of one of the cones

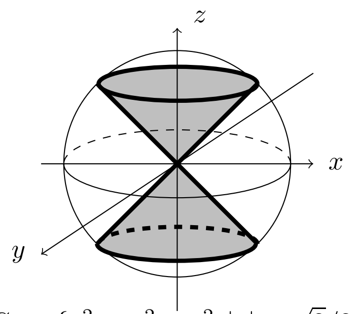
$$C_1 = \{\vec{\mathbf{r}} \in \mathbb{R}^3; z^2 \geq x^2 + y^2, |z| < \sqrt{2}/2\}, \qquad C_2 = \{\vec{\mathbf{r}} \in \mathbb{R}^3; z^2 \geq x^2 + y^2, |z| < 1\}$$

which are not those needed for the exercise, as shown in the figure below. Both C_1 and C_2 have flat bases, while $C \cap B$ has spherical caps. The volume of C_1 and C_2 can easily be computed using cylindrical coordinates, while that of $C \cap B$ using spherical coordinates.

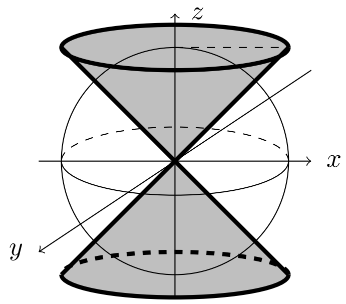
Moreover, a few people forgot the lower half of the cone: note that in the definition of C there is no condition on the sign of z.



$C \cap B$



$C_1 = \{z^2 \geq x^2 + y^2, |z| < \sqrt{2}/2\}$



$C_2 = \{z^2 \geq x^2 + y^2, |z| < 1\}$

Exercise 2: This is the exercise where I found most errors.

- Several students did not try to prove the "if and only if" statement, but proved only one implication, namely that solenoidal fields whose components are Helmholtz solutions are solutions of Maxwell equation.
- The key point of the implication "if $\vec{\mathbf{F}}$ is Maxwell solution than is solenoidal and its component are Helmholtz solutions" consists in verifying that
$$\vec{\nabla} \cdot \vec{\mathbf{F}} = \vec{\nabla} \cdot \Big(\frac{1}{k^2}\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})\Big) = \frac{1}{k^2}\vec{\nabla} \cdot \Big(\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}})\Big) = 0$$
because of the divergence of a curl is zero (identity (25)). This is the point everybody missed.
- Some tried to prove the converse implication by contradiction, namely by showing that different cases (i.e. $\vec{\mathbf{F}}$ solenoidal but not Helmholtz solution, Helmholtz solution but not solenoidal, neither) lead to show that $\vec{\mathbf{F}}$ is not solution to Maxwell equation. This approach might work, but is more complicated than the direct proof above.
- The fact that a field $\vec{\mathbf{F}}$ is not solenoidal does not imply that $\vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{F}})$ is non-zero. For example $\vec{\mathbf{F}} = \vec{\mathbf{r}}$ satisfies $\vec{\nabla} \cdot \vec{\mathbf{F}} = 3$ and thus $\vec{\nabla}(\vec{\nabla} \cdot \vec{\mathbf{F}}) = \vec{\mathbf{0}}$.
- In proofs like this one you need to write *clearly* what you assume and what you try to prove in each part of the solution. In some cases, it was not possible to understand which one of the two implications students were trying to prove.

Exercise 3:

- Several people tried to compute the line integral of $\vec{\mathbf{G}} \cdot \hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the planar unit vector normal to ∂R as in (97). This was not what was requested. The question asked for "the line integral of $\vec{\mathbf{G}}$", which means the line integral of its tangential component, as defined in (44), not the integral of the normal component.
- The "log" in the question is a natural logarithm, as usual in maths. Who used basis-10 logarithm did not lose marks.
- Green's theorem is not Green's function! And both deserve a capital letter.