Vector calculus MA3VC 2016–17: Assignment 3

MA3VC: Part 3 students only.
Handed out: Thursday 1st December. Due: **Thursday 8th December, 12 noon**.
You can use formulas and identities from the lecture notes. Do not use red pen nor pencil.
Marking will be anonymous, so please write your name only on the "assessed work coversheet" and not on your work. Write your student number both on the back of the coversheet and each page of your work.
Total marks: 25. (10% of the total marks for the module.)

(Exercise 1 — 7 marks) Compute the volume of the portion D of the unit ball $B = \{\vec{\mathbf{r}} \in \mathbb{R}^3; |\vec{\mathbf{r}}| < 1\}$ that is not contained in the double cone $C = \{\vec{\mathbf{r}} \in \mathbb{R}^3; z^2 \geq x^2 + y^2\}$.
Hint: Use a set of special coordinates.

(Exercise 2 — 10 marks) Fix a positive number k. Consider the so called "time-harmonic Maxwell equation" for a vector field $\vec{\mathbf{F}}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathbf{F}}) - k^2 \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

and the "Helmholtz equation" for a scalar field f

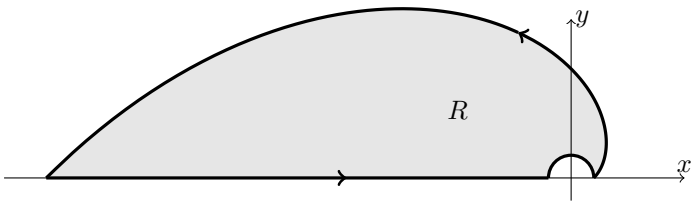
$$\Delta f + k^2 f = 0.$$

Prove that a smooth vector field $\vec{\mathbf{F}}$ satisfies Maxwell's equation *if and only if* it is solenoidal and its three components satisfy the Helmholtz equation.
Hint: Recall the identities proved in Section 1.4 of the notes.

(Exercise 3 — 8 marks) Compute the line integral of $\vec{\mathbf{G}} = -(x + y + x^3 + y^2)\hat{\boldsymbol{\imath}} + (x + y - 2xy)\hat{\boldsymbol{\jmath}}$ along the boundary ∂R of the region R depicted in the figure below, which is defined in polar coordinates by

$$R = \big\{\vec{\mathbf{r}} \in \mathbb{R}^2,\ 0 < \log r < \theta < \pi\big\}.$$

Assume that the path ∂R is oriented in the anticlockwise direction.
Hint: Do not try to compute the integral by brute force; use instead some important theorem.



Please check carefully the list of common errors on page 110 of the notes and try not to commit them!