MA2VC, Vector Calculus, Assignment 1

due: 12pm, 2 Nov 2012 (late assignments will not be accepted, and marks will be deducted for poor presentation)

1) (4 marks) Sketch the volume defined by

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad \text{and} \quad 3x + 2y + z \leq 6$$

and determine the outward pointing unit normal, $\hat{\mathbf{n}}$, on each of the four sides, by taking the gradient of a scalar field defined such that the side is a level surface, *i.e.*, $\phi(\mathbf{r}) = \text{constant}$.

2) (4 marks) Prove the vector differential identity

$$\nabla \cdot (f(\nabla g \times \nabla h)) = \nabla f \cdot (\nabla g \times \nabla h)$$

You can make use of the identities on the reverse side.

3) (4 marks) Demonstrate that the above identity is satisfied for the scalar fields

$$\begin{aligned} f(\mathbf{r}) &= e^{x+y} \\ g(\mathbf{r}) &= xy + z^2 \\ h(\mathbf{r}) &= e^z \end{aligned}$$

4) (4 marks) Show that the vector field

$$\mathbf{F}(\mathbf{r}) = 2xz\hat{\mathbf{i}} + 2yz\hat{\mathbf{j}} + (x^2 + y^2 + z^2)\hat{\mathbf{k}}$$

is irrotational and find a scalar potential, $\phi(\mathbf{r})$.

5) (4 marks) Show that the vector field

$$\mathbf{F}(\mathbf{r}) = 2xy\hat{\mathbf{i}} - y^2\hat{\mathbf{j}}$$

is solenoidal and find a vector potential, $\mathbf{A}(\mathbf{r})$. Hint: look for one that just has a z component.

Vector Differential Identities

(a) $\nabla(fg) = f\nabla g + g\nabla f$

(b) $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\nabla \cdot \mathbf{F})\mathbf{G} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

(f) $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

(g) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(h) $\nabla \times (\nabla f) = 0$

(i) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$