Vector calculus MA2VC 2013–14 — Assignment 2 SOLUTIONS

(1) We compute the total derivative of $\vec{\mathbf{a}}$

$$\frac{d\vec{\mathbf{a}}}{dt}(t) = \frac{d(e^t\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + e^{2t}\cos t\hat{\boldsymbol{k}})}{dt} = e^t\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + e^{2t}(2\cos t - \sin t)\hat{\boldsymbol{k}};$$

the gradient of f

$$\vec{\nabla} f = \frac{\partial(\sqrt{x^2+y^2+z^2})}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial(\sqrt{x^2+y^2+z^2})}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial(\sqrt{x^2+y^2+z^2})}{\partial z}\hat{\boldsymbol{k}} = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|};$$

and their scalar product, which (thanks to the chain rule (34)) is the total derivative of the compound function:

$$\begin{aligned}
\frac{d\big(f(\vec{\mathbf{a}})\big)}{dt}(t) &= \vec{\nabla} f\big(\vec{\mathbf{a}}(t)\big) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t) \qquad\qquad \text{(chain rule)}\\
&= \frac{xe^t + y + ze^{2t}(2\cos t - \sin t)}{\sqrt{x^2+y^2+z^2}}\\
&= \frac{e^{2t} + t + e^{4t}(2\cos^2 t - \cos t\sin t)}{\sqrt{e^{2t} + t^2 + e^{4t}\cos^2 t}}.
\end{aligned}$$

In the last step we have substituted the values of the curve components $a_1(t)$, $a_2(t)$ and $a_3(t)$ in place of x, y and z, so that we have a function of t only.

Alternatively, $\frac{d(f(\vec{\mathbf{a}}))}{dt}(t)$ can be found by computing and deriving the compound function:

$$\begin{aligned}
f\big(\vec{\mathbf{a}}(t)\big) &= \sqrt{e^{2t} + t^2 + e^{4t}\cos^2 t},\\
\frac{d\big(f(\vec{\mathbf{a}})\big)}{dt}(t) &= \frac{2e^{2t} + 2t + 4e^{4t}\cos^2 t - 2e^{4t}\cos t\sin t}{2\sqrt{e^{2t} + t^2 + e^{4t}\cos^2 t}} = \frac{e^{2t} + t + e^{4t}(2\cos^2 t - \cos t\sin t)}{\sqrt{e^{2t} + t^2 + e^{4t}\cos^2 t}}.
\end{aligned}$$

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(2) As described in Section 2.1.1 of the notes, the length of a path can be measured by integrating the constant scalar field $f(\vec{\mathbf{r}}) = 1$. We compute the total derivative of $\vec{\mathbf{b}}$

$$\begin{aligned}
\vec{\mathbf{b}} &= 5\cos t\,\hat{\boldsymbol{\imath}} + \big(3\cos t - 4\sqrt{2}\sin t\big)\hat{\boldsymbol{\jmath}} + \big(4\cos t + 3\sqrt{2}\sin t\big)\hat{\boldsymbol{k}};\\
\frac{d\vec{\mathbf{b}}}{dt} &= -5\sin t\,\hat{\boldsymbol{\imath}} + \big(-3\sin t - 4\sqrt{2}\cos t\big)\hat{\boldsymbol{\jmath}} + \big(-4\sin t + 3\sqrt{2}\cos t\big)\hat{\boldsymbol{k}};
\end{aligned}$$

its magnitude (which enters the definition of the infinitesimal length element $\mathrm{d}s$)

$$\begin{aligned}
\left|\frac{d\vec{\mathbf{b}}}{dt}\right|^2 &= 25\sin^2 t + \big(-3\sin t - 4\sqrt{2}\cos t\big)^2 + \big(-4\sin t + 3\sqrt{2}\cos t\big)^2\\
&= 25\sin^2 t + \big(9\sin^2 t + 32\cos^2 t + 24\sqrt{2}\sin t\cos t\big)\\
&\qquad + \big(16\sin^2 t + 18\cos^2 t - 24\sqrt{2}\sin t\cos t\big)\\
&= 50\sin^2 t + 50\cos^2 t\\
&= 50;
\end{aligned}$$

and the integral

$$\mathrm{Length}(\Gamma) = \int_\Gamma 1\,\mathrm{d}t = \int_0^\pi \left|\frac{d\vec{\mathbf{b}}}{dt}\right| \mathrm{d}t = \int_0^\pi \sqrt{50}\;\mathrm{d}t = \pi\sqrt{50} = 5\pi\sqrt{2} \approx 22.2.$$

Since the magnitude of $\frac{d\vec{\mathbf{b}}}{dt}$ is constant ($|\frac{d\vec{\mathbf{b}}}{dt}| = \sqrt{50}$), the integration turns out to be very easy. (The path Γ is half circle of radius $\sqrt{50}$ and $\vec{\mathbf{a}}$ travels on it with constant speed.)

(3) First, we verify that the two paths (which we denote Γ_c and Γ_d) share the endpoints:

$$\vec{\mathbf{c}}\Big(-\frac{\pi}{2}\Big) = \vec{\mathbf{d}}(-1) = -\hat{\boldsymbol{\imath}}, \qquad \vec{\mathbf{c}}\Big(\frac{\pi}{2}\Big) = \vec{\mathbf{d}}(1) = \hat{\boldsymbol{\imath}};$$

they both start at $-\hat{\boldsymbol{\imath}}$ and end at $\hat{\boldsymbol{\imath}}$. Then, we compute the total derivatives of the curves

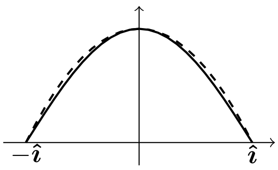
$$\frac{d\vec{\mathbf{c}}}{dt} = \frac{2}{\pi}\hat{\boldsymbol{\imath}} - \sin t\,\hat{\boldsymbol{\jmath}}, \qquad \frac{d\vec{\mathbf{d}}}{d\tau} = \hat{\boldsymbol{\imath}} - 2\tau\hat{\boldsymbol{\jmath}},$$

and use them to compute the integrals of $\vec{\mathbf{G}} = y\hat{\boldsymbol{\imath}}$ along the two paths:

$$\begin{aligned}
\int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vec{\mathbf{G}}\big(\vec{\mathbf{c}}(t)\big) \cdot \frac{d\vec{\mathbf{c}}}{dt}\,\mathrm{d}t \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (y\hat{\boldsymbol{\imath}}) \cdot \Big(\frac{2}{\pi}\hat{\boldsymbol{\imath}} - \sin t\hat{\boldsymbol{\jmath}}\Big)\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t\hat{\boldsymbol{\imath}}) \cdot \Big(\frac{2}{\pi}\hat{\boldsymbol{\imath}} - \sin t\hat{\boldsymbol{\jmath}}\Big)\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,\mathrm{d}t = \frac{2}{\pi}\sin t\Big|_{t=-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{4}{\pi} \approx 1.273; \\
\int_{\Gamma_d} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{-1}^{1} \vec{\mathbf{G}}\big(\vec{\mathbf{d}}(\tau)\big) \cdot \frac{d\vec{\mathbf{d}}}{d\tau}\,\mathrm{d}\tau \\
&= \int_{-1}^{1} (y\hat{\boldsymbol{\imath}}) \cdot (\hat{\boldsymbol{\imath}} - 2\tau\hat{\boldsymbol{\jmath}})\,\mathrm{d}\tau = \int_{-1}^{1} (1-\tau^2)\hat{\boldsymbol{\imath}} \cdot (\hat{\boldsymbol{\imath}} - 2\tau\hat{\boldsymbol{\jmath}})\,\mathrm{d}\tau \\
&= \int_{-1}^{1} (1-\tau^2)\,\mathrm{d}\tau = \Big(\tau - \frac{1}{3}\tau^3\Big)\Big|_{\tau=-1}^{1} = \frac{4}{3} \approx 1.333.
\end{aligned}$$

We note that they give different values: $\int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \frac{4}{\pi} \neq \frac{4}{3} = \int_{\Gamma_d} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}}$. (The values are actually quite close to each other since the paths are quite similar, see Figure **??**, where $\vec{\mathbf{c}}$ is plotted as a continuous line and $\vec{\mathbf{d}}$ is dashed.) Since two line integrals along different paths *sharing the endpoints* are different, by (the contrapositive of) the fundamental theorem of vector calculus (Section 2.1.3 in the notes), we have that $\vec{\mathbf{G}}$ is not a gradient, so it is not conservative.

The easiest proof of the fact that $\vec{\mathbf{G}}$ is non-conservative, consists in noting that $\vec{\nabla}\times\vec{\mathbf{G}} = -\hat{\boldsymbol{k}} \neq \vec{\mathbf{0}}$, thus $\vec{\mathbf{G}}$ is not irrotational and the assertion follows from the contrapositive of the implication "conservative⇒irrotational" in the box in Section 1.5.



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(4) For $n = 1, 2, 3$, the nth component of $\vec{\mathbf{H}}$ depends on the nth variable only, thus, by the definition of the curl operator, $\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{H}}$ is irrotational (equivalently, we compute the curl and verify directly that $\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{0}}$).

We compute a scalar potential φ by proceeding as in Exercise 1.42:

$$\begin{aligned}
&\vec{\mathbf{H}} = \vec{\nabla}\varphi \quad \Rightarrow \quad \frac{\partial\varphi(x,y,z)}{\partial x} = H_1 = x \quad \Rightarrow \quad \varphi(x,y,z) = \frac{1}{2}x^2 + g(y,z), \\
&\frac{\partial\varphi(x,y,z)}{\partial y} = \frac{\partial\big(\frac{1}{2}x^2 + g(y,z)\big)}{\partial y} = \frac{\partial g(y,z)}{\partial y} = H_2 = y^2 \quad \Rightarrow \quad g(y,z) = \frac{1}{3}y^3 + f(z), \\
&\frac{\partial\varphi(x,y,z)}{\partial z} = \frac{\partial\big(\frac{1}{2}x^2 + \frac{1}{3}y^3 + f(z)\big)}{\partial z} = \frac{\partial f(z)}{\partial z} = H_3 = z^3 \quad \Rightarrow \quad f(z) = \frac{1}{4}z^4 + \lambda, \\
&\Rightarrow \varphi(x,y,z) = \frac{1}{2}x^2 + g(y,z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + f(z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{4}z^4 + \lambda, \qquad \forall\lambda \in \mathbb{R}.
\end{aligned}$$

Both paths of Exercise **(3)** have endpoints $\vec{\mathbf{c}}(-\frac{\pi}{2}) = \vec{\mathbf{d}}(-1) = -\hat{\boldsymbol{\imath}}$ and $\vec{\mathbf{c}}(\frac{\pi}{2}) = \vec{\mathbf{d}}(1) = \hat{\boldsymbol{\imath}}$. Since $\vec{\mathbf{H}}$ is conservative, by the fundamental theorem of vector calculus, the integral along any path connecting $-\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\imath}}$ can be computed as:

$$\int_{\Gamma_c} \vec{\mathbf{H}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_c} \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_d} \vec{\mathbf{H}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_d} \vec{\nabla}\varphi \cdot \mathrm{d}\vec{\mathbf{r}} = \varphi(\hat{\boldsymbol{\imath}}) - \varphi(-\hat{\boldsymbol{\imath}}) = \frac{1}{2} - \frac{1}{2}(-1)^2 = 0.$$