Vector calculus MA2VC 2013–14 — Assignment 3 SOLUTIONS

(1a) We can use the following change of variables (we write here also the inverse): [1]

$$(I)\ \begin{cases}\xi = xy,\\ \eta = y,\end{cases}\qquad \begin{pmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{pmatrix} = \begin{pmatrix} y & x\\ 0 & 1\end{pmatrix},\qquad \frac{\partial(\xi,\eta)}{\partial(x,y)} = y,\qquad \begin{cases}x = \frac{\xi}{\eta},\\ y=\eta,\end{cases}\qquad \frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{\eta}.$$

(1b) The integral is computed as:

$$\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta \overset{(50)}{=} \iint_S \frac{\xi(x,y)}{\eta(x,y)}\left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right|\mathrm{d}x\,\mathrm{d}y = \iint_S \frac{xy}{y}\,y\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 x\,\mathrm{d}x\int_0^1 y\,\mathrm{d}y = \frac{1}{2}\frac{1}{2} = \frac{1}{4}.$$

We can double check the computation above using the iterated integral on T:

$$\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1\int_0^\eta \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_0^1 \frac{\eta^2}{2\eta}\,\mathrm{d}\eta = \frac{1}{4}.$$

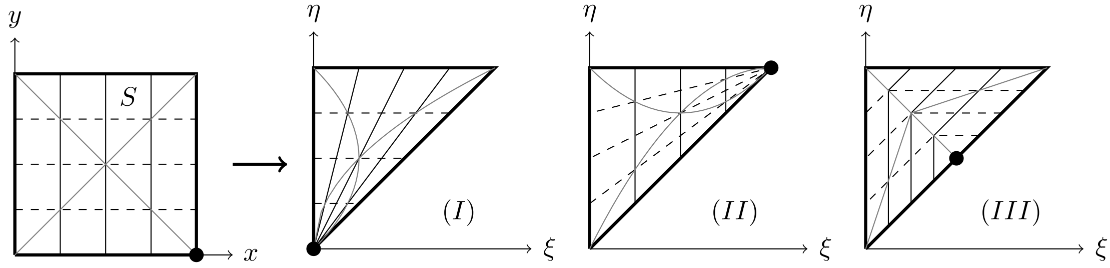

Figure 1: *The square S and its image T under the three transformation described in exercise **(1)**; many other options are possible. In each case, the point $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}$ are mapped into $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\xi}}+\hat{\boldsymbol{\eta}}$, respectively, while $\hat{\boldsymbol{\imath}}$ is mapped into the point denoted with the little circle. The continuous, dashed and grey lines represent the images of the vertical, horizontal and diagonal segments in the first plot. Note that the first two transformations (which are polynomials of degree two) map some straight lines into parabolas, while the third one (which is piecewise affine) maps straight lines into polygonal lines.*

[1] How did we find this transformation? First we decide how to change the domain geometrically: since we are transforming a square into a triangle, have to "get rid of" one of the sides of S, we *choose* to maintain the upper and the left side, collapse the lower one and transform the right one into the diagonal (see below for other options). We then note that we can maintain the vertical coordinate of every point and modify only the horizontal one, so we fix $\eta = y$. We want to modify x by multiplying with a suitable function, so $\xi = xg(x,y)$. The function g must have value 1 on the horizontal line $y = 1$ (so the upper side is preserved), and value 0 on the horizontal line $y = 0$ (so the lower side is collapsed). The choice $g(x,y) = y$ is the easiest that satisfies these two conditions ($g(x,1) = 1, g(x,0) = 0$).

Then, we can immediately verify that, under this transformation, the upper side of S is preserved ($y = 1 \Rightarrow \eta = 1$), the left side is preserved ($x = 0 \Rightarrow \xi = 0$), the right side is mapped into the diagonal side of T ($x = 1 \Rightarrow \xi = \eta$), the lower side is collapsed to the origin ($y = 0 \Rightarrow \xi = \eta = 0$). Thus the transformation maps correctly the boundaries: $\partial S \to \partial T$. It is surjective (for all $\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}} \in T$, $\xi/\eta\hat{\boldsymbol{\imath}} + \eta\hat{\boldsymbol{\jmath}}$ belongs to S), and it is injective in the *interior* of S:

$$\vec{\mathbf{T}}(x_A,y_A) = x_Ay_A\hat{\boldsymbol{\xi}} + y_A\hat{\boldsymbol{\eta}} = x_By_B\hat{\boldsymbol{\xi}} + y_B\hat{\boldsymbol{\eta}} = \vec{\mathbf{T}}(x_B,y_B)\quad\Rightarrow\quad \begin{matrix}x_Ay_A = x_By_B,\\ y_A = y_B,\end{matrix}\quad\overset{y_A,y_B>0}{\Rightarrow}\quad \begin{matrix}x_A = x_B,\\ y_A = y_B.\end{matrix}$$

(The verification of the mapping of the boundary and the bijectivity was not required in the exercise, however it is what you should do to check if your transformation is correct.)

A smart way (from one of your solutions) to obtain and verify the transformation, is to rewrite the domains as

$$T = \{0 < \xi < \eta < 1\} = \{0 < \xi/\eta < 1,\ 0 < \eta < 1\},\qquad S = \{0 < x < 1,\ 0 < y < 1\},$$

from which the inverse transformation $x = \frac{\xi}{\eta}$, $y = \eta$ is immediate (this corresponds to the setting of Example 2.22).

Many other changes of variables are possible, some simple ones are

$$(II)\ \begin{cases}\xi = x,\\ \eta = x + y - xy,\end{cases}\qquad (III)\ \begin{cases}\begin{cases}\xi = \frac{1}{2}x,\\ \eta = y + \frac{1}{2}x,\end{cases} & \text{for } x+y\le 1,\\ \begin{cases}\xi = x + \frac{1}{2}(y-1),\\ \eta = \frac{1}{2}(y+1),\end{cases} & \text{for } x+y>1.\end{cases}$$

The transformation *(II)* corresponds to that described in Example 2.21 for general y-simple domain; *(III)* is "piecewise affine" (and continuous). See a representation of the three transformations in Figure 1; you can play around with these transformations using `VCplotter.m`. Note that the transformation of the unit cube into a tetrahedron on page 43 on the notes is very similar to this exercise.

(2) The first step to solve the problem is to draw a sketch of the domain D, see left plot in Figure 2. Since D is defined by $x^2 + y^2 < z < \sqrt{x^2 + y^2}$, we have, for all points in D, $x^2 + y^2 < 1$ and $0 < z < 1$ (this can also be seen by intersecting the two surfaces defining the boundary of D).

The most natural coordinate system to describe D is the cylindrical system, as the domain is defined using z and $r = \sqrt{x^2 + y^2}$. We can also use spherical coordinates, but this makes the computation a bit harder. The use of Cartesian coordinates leads to a long and complicated computation. Several ways to solve the problem are possible.

(Version i) In cylindrical coordinates, the domain reads $D = \{r^2 < z < r\}$. Its volume is:

$$\mathrm{Vol}(D) = \iiint_D \mathrm{d}V \overset{(67)}{=} \iiint_D r \,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}z = \int_{-\pi}^{\pi} \bigg(\int_0^1 \Big(\int_{r^2}^{r} \mathrm{d}z \Big) r \,\mathrm{d}r \bigg) \mathrm{d}\theta$$
$$= 2\pi \int_0^1 (r - r^2) r \,\mathrm{d}r = 2\pi \Big(\frac{1}{3} - \frac{1}{4} \Big) = \frac{\pi}{6} \approx 0.523.$$

(Version ii) Equivalently, the domain can also be written as $D = \{z < r < \sqrt{z}\}$:

$$\mathrm{Vol}(D) = \iiint_D \mathrm{d}V \overset{(67)}{=} \int_{-\pi}^{\pi} \bigg(\int_0^1 \Big(\int_z^{\sqrt{z}} r \,\mathrm{d}r \Big) \mathrm{d}z \bigg) \mathrm{d}\theta = 2\pi \int_0^1 \frac{z - z^2}{2} \,\mathrm{d}z = 2\pi \Big(\frac{1}{4} - \frac{1}{6} \Big) = \frac{\pi}{6}.$$

(Version iii) Alternatively, we note that D is the set difference of two solids of revolution as those in equation (68): D is the difference of the volumes $r(z) < \sqrt{z}$ (the outer paraboloid) and $r(z) \le z$ (the inner cone), both defined for $z \in (0, 1)$. Thus, we can use (twice) formula (69) for the volume of a solid of revolution and subtract the two results obtained. So

$$\mathrm{Vol}(D) = \mathrm{Vol}(\text{paraboloid}) - \mathrm{Vol}(\text{cone}) \overset{(69)}{=} \pi \int_0^1 (\sqrt{z})^2 \,\mathrm{d}z - \pi \int_0^1 z^2 \,\mathrm{d}z = \pi \frac{1}{2} - \pi \frac{1}{3} = \frac{\pi}{6}.$$

(Version iv) The computation in spherical coordinates is more complicated. Using the relations (73) between spherical and cylindrical coordinates, we see that the cone complement $z < r$ is translated into $\phi > \frac{\pi}{4}$; the condition $z > 0$ into $\phi < \frac{\pi}{2}$; the paraboloid $z > r^2$ in $\rho < \frac{\cos\phi}{\sin^2\phi}$ (verify where these inequalities come from!). Thus the domain can be described in this system as

$$D = \Big\{ \vec{\mathbf{r}} \in \mathbb{R}^3, \text{ s.t. } \frac{\pi}{4} < \phi < \frac{\pi}{2},\ \rho < \frac{\cos\phi}{\sin^2\phi} \Big\},$$

and its volume is

$$\mathrm{Vol}(D) = \iiint_D \mathrm{d}V \overset{(74)}{=} \iiint_D \rho^2 \sin\phi \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\theta = \int_{-\pi}^{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \int_0^{\frac{\cos\phi}{\sin^2\phi}} \rho^2 \,\mathrm{d}\rho \,\mathrm{d}\phi \,\mathrm{d}\theta$$
$$= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi \frac{1}{3} \frac{\cos^3\phi}{\sin^6\phi} \,\mathrm{d}\phi$$
$$= \frac{2}{3}\pi \frac{-\cot^4\phi}{4} \bigg|_{\phi=\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{6}. \qquad \Big(\cot\phi = \frac{\cos\phi}{\sin\phi} \Big)$$

(Version v) The domain can be seen as the part of space between the graph of the scalar fields $g_{bot}(r, \theta) = r^2$ and $g_{top}(r, \theta) = r$, expressed in polar coordinates (here we use two-dimensional polar coordinates, as opposed to three-dimensional cylindrical coordinates). So the volume is the difference between the *double* integrals of the two field over the unit disc:

$$\mathrm{Vol}(D) = \iint_{\{r\hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}, r<1\}} (g_{top} - g_{bot}) \,\mathrm{d}A = \int_{-\pi}^{\pi} \bigg(\int_0^1 (r - r^2) r \,\mathrm{d}r \bigg) \mathrm{d}\theta = 2\pi \Big(\frac{1}{3} - \frac{1}{4} \Big) = \frac{\pi}{6}.$$

(In summary, the volume can be computed either as triple integral of 1, versions *(i–iv)*, or double integral of some non-constant field, version *(v)*.)

(3) We first write the triangle R as the graph of a two dimensional field g. Since R is a subset of a plane, g is affine, i.e. polynomial of degree one: $g(x,y) = a + bx + cy$. We compute the coefficients a, b, c: R must contain the point $\hat{\boldsymbol{k}}$, so $1 = g(0,0) = a$; the point $3\hat{\boldsymbol{\imath}}$ so $0 = g(3,0) = 1 + bx$ and $b = -\frac{1}{3}$; the point $2\hat{\boldsymbol{\jmath}}$ so $0 = g(0,2) = 1 + cy$ and $c = -\frac{1}{2}$. In summary, $g(x,y) = 1 - \frac{1}{3}x - \frac{1}{2}y$.

The triangle R is the graph of g on the triangular domain

$$\widetilde{R} = \big\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ 0 < x < 3 - 3y/2,\ 0 < y < 2\big\}, \quad \text{i.e.} \quad R = \big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \widetilde{R}, z = g(x,y)\big\}.$$

$\widetilde{R}$ is the projection of R on the xy-plane and has vertices $3\hat{\boldsymbol{\imath}}$, $2\hat{\boldsymbol{\jmath}}$ and $\vec{\mathbf{0}}$.

The flux of $\vec{\mathbf{F}}$ through R is defined as the integral $\iint_R \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S$. Since R is a graph surface, we do not need to compute the unit normal field (which, by the way, is $\hat{\boldsymbol{n}} = \frac{2}{7}\hat{\boldsymbol{\imath}} + \frac{3}{7}\hat{\boldsymbol{\jmath}} + \frac{6}{7}\hat{\boldsymbol{k}}$, as obtained from the gradient of $z - g(x,y)$ or from (57)), but we can simply use formula (59) in the notes:

$$\begin{aligned}
\iint_R \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S &\overset{(59)}{=} \iint_{\widetilde{R}} \Big(-F_1 \frac{\partial g}{\partial x} - F_2 \frac{\partial g}{\partial y} + F_3 \Big)\, \mathrm{d}A \\
&= \iint_{\widetilde{R}} \Big(x\frac{1}{3} + y\frac{1}{2} + \underbrace{z}_{=g(x,y)} \Big)\, \mathrm{d}A \\
&= \iint_{\widetilde{R}} \Big(x\frac{1}{3} + y\frac{1}{2} + 1 - \frac{1}{3}x - \frac{1}{2}y \Big)\, \mathrm{d}A \\
&= \int_0^2 \Big(\int_0^{3-\frac{3}{2}y} 1\, \mathrm{d}x \Big)\, \mathrm{d}y = \int_0^2 \Big(3 - \frac{3}{2}y \Big)\, \mathrm{d}y = \Big(3y - \frac{3}{4}y^2 \Big)\Big|_0^2 = 3.
\end{aligned}$$

A different way to solve the exercise is to note that the scalar product $\vec{\mathbf{r}} \cdot \hat{\boldsymbol{n}}$ is constant on R: $\vec{\mathbf{r}} \cdot \hat{\boldsymbol{n}}(\vec{\mathbf{r}}) = \frac{6}{7}$. Therefore, the flux is

$$\iint_R \vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S = (\vec{\mathbf{r}} \cdot \hat{\boldsymbol{n}})\mathrm{Area}(R) = \frac{6}{7} \frac{|(3\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{k}}) \times (2\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})|}{2} = \frac{6}{7} \frac{|2\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}|}{2} = \frac{6}{7}\frac{7}{2} = 3,$$

as the area of a triangle is half the magnitude of the vector product of two of its edges (recall that the magnitude of a vector product is the area of a parallelogram).

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(4) Every expression contains a mistake:

(a) contains the curl of a scalar field, which is not defined.

In **(b)**, the first two terms are vector quantities, while the integral is scalar, so they can not be summed to each other.

(c) contains the expression $\hat{\boldsymbol{\imath}}$(scalar quantity)$\vec{\nabla} f$, which makes no sense.

(d) is tricky: it contains a product in the form $\vec{\mathbf{F}} \times \vec{\mathbf{u}} \times \vec{\mathbf{w}}$, which is ambiguous as the vector product is not associative ($(\vec{\mathbf{F}} \times \vec{\mathbf{u}}) \times \vec{\mathbf{w}}$ and $\vec{\mathbf{F}} \times (\vec{\mathbf{u}} \times \vec{\mathbf{w}})$ are in general not equal to each other, so we do not know how to give a clear meaning to the product without brackets).

In **(e)**, the last occurrence of the vector field $\vec{\mathbf{F}}$ (which should receive and give vector values) is evaluated in a scalar value.

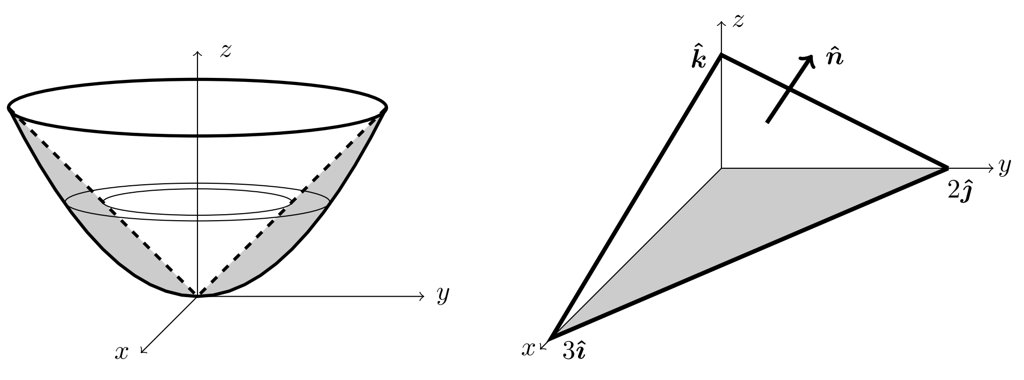

Figure 2: *Left: the domain D of exercise **(2)** (D is obtained by a rotation of the shaded part). Right: the triangle R of exercise **(3)** (thick line) and its projection $\widetilde{R}$ on the xy-plane (shaded).*