Vector calculus MA2VC 2013–14 — Assignment 3

Handed out: Friday 15th November. Due: **Friday 29th November, 12 noon**.
Late assignments will not be accepted. Do not use red pen or pencil.
You can use formulas and identities from the lecture notes. Drawing a sketch of the domains may help you.
Total marks: 20. (5% of the total marks for the module.)

(1) (5 marks) Consider the unit square $S = (0,1)^2 = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ 0 < x < 1,\ 0 < y < 1\}$ and the triangle $T = \{\xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}},\ 0 < \xi < \eta < 1\}$ with vertices $\vec{\mathbf{0}}$, $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\eta}}$.

(1a) Find a simple (bijective) change of variables $(x, y) \mapsto (\xi, \eta)$ that maps S into T. Hint: consider a polynomial transformation that deforms the x variable only; in the notes you can find a (more complicated) example of a similar transformation.

(1b) Use the change of variables you found to compute

$$\iint_T \frac{\xi}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

(2) (5 marks) Compute the volume of the domain D bounded by the unit cone $\{x^2+y^2 = z^2\}$ and the paraboloid $\{z = x^2 + y^2\}$, i.e.:

$$D = \Big\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ \text{s.t. } x^2 + y^2 < z < \sqrt{x^2+y^2}\Big\}.$$

Hint: find in Section 2.3 of the lecture notes a suitable system of coordinates to describe D.

(3) (5 marks) Consider the triangle R with vertices $3\hat{\boldsymbol{\imath}}$, $2\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. Fix on R the orientation $\hat{\boldsymbol{n}}$ such that $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} > 0$. Compute the flux of the vector field $\vec{\mathbf{F}} = \vec{\mathbf{r}}$ through R.
Hint: consider R as the surface graph of a two-dimensional scalar field $g(x, y)$.
Recall that, since R lies in a plane, the field g must be an affine function.

(4) (5 marks) State which of the following five expressions are well-defined and unambiguous; for those that are not, state precisely what does not work.
Consider only the notation used in the lecture notes and in class. Here $\vec{\mathbf{F}}, \vec{\mathbf{G}}, \vec{\mathbf{H}} : \mathbb{R}^3 \to \mathbb{R}^3$ are smooth vector fields; $f, g : \mathbb{R}^3 \to \mathbb{R}$ are smooth scalar fields; $D \subset \mathbb{R}^3$ is a domain; Γ is an oriented path; $\vec{\mathbf{a}}, \vec{\mathbf{b}} : \mathbb{R} \to \mathbb{R}^3$ are smooth curves. (As usual, in equations *(a)–(d)* all fields are evaluated in $\vec{\mathbf{r}}$, even if $\vec{\mathbf{r}}$ is not explicitly written; e.g. $\vec{\mathbf{F}} = \vec{\mathbf{F}}(\vec{\mathbf{r}})$.)

$$(a)\quad (\vec{\mathbf{r}} - |\vec{\mathbf{F}}|\vec{\mathbf{F}}) \cdot \bigg(\vec{\nabla} \times \Big((F_1 + |\vec{\mathbf{F}}|^2 + F_3^3)\vec{\nabla} \cdot (\cos x\vec{\mathbf{F}} + \sin y\vec{\mathbf{G}} - 4\hat{\boldsymbol{\jmath}})\Big) + \vec{\nabla}\Big(fF_2 + e^g\frac{\partial F_1}{\partial z}\Big)\bigg),$$

$$(b)\quad \vec{\nabla}\Delta f + \bigg(\Big(\vec{\nabla} \times \vec{\mathbf{F}} + \hat{\boldsymbol{\imath}}\sqrt{|\vec{\mathbf{F}}|}\Big) \cdot \vec{\nabla}\bigg)\vec{\mathbf{F}} - 5\int_\Gamma \Big(\vec{\mathbf{F}} \times (\vec{\nabla} \times \vec{\mathbf{F}}) + \vec{\nabla}\frac{\partial f}{\partial y} - \vec{\boldsymbol{\Delta}}\vec{\mathbf{F}}\Big) \cdot \mathrm{d}\vec{\mathbf{r}},$$

$$(c)\quad \iiint_D \Big(\hat{\boldsymbol{\imath}}(\vec{\mathbf{F}} \cdot \vec{\mathbf{G}} \times \vec{\mathbf{H}} + \cosh(\vec{\mathbf{F}} \cdot \vec{\mathbf{H}} - F_1H_2 - F_2H_1))\vec{\nabla} f\Big) \cdot \vec{\mathbf{r}}\,\mathrm{d}V + e^{\frac{1}{\pi}\sqrt{\iiint_D \mathrm{d}V}},$$

$$(d)\quad \vec{\mathbf{F}} \times \Big(\hat{\boldsymbol{\imath}}\cos 3 + |\vec{\mathbf{F}} - 9\hat{\boldsymbol{\imath}}|^4\hat{\boldsymbol{\jmath}} \times \big(\vec{\mathbf{G}} \cdot (\vec{\mathbf{H}} - \hat{\boldsymbol{\imath}})\big)\vec{\mathbf{F}}\Big) \times (\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) + (\vec{\mathbf{F}} \cdot \vec{\mathbf{G}})\vec{\mathbf{F}} \times \vec{\mathbf{G}};$$

in equation *(e)* the dependences of <u>all</u> fields and curves are written explicitly ($t \in \mathbb{R}$):

$$(e)\quad f\Big(\vec{\mathbf{F}}(7\vec{\mathbf{a}}(t) \times \vec{\mathbf{b}}(t))\Big)e^{\vec{\mathbf{a}}(t)\cdot\hat{\boldsymbol{\imath}} - a_2(t)}\vec{\mathbf{G}}\bigg(\vec{\mathbf{a}}\Big(\vec{\mathbf{G}}\Big(\frac{d\vec{\mathbf{b}}}{dt}(t)\Big) \cdot \vec{\mathbf{H}}(\vec{\mathbf{b}}(t))\Big)\bigg) + \vec{\mathbf{F}}\Big(e^{|\vec{\mathbf{b}}(t)|^2}|\vec{\mathbf{a}}(t) \times \hat{\boldsymbol{k}}t|\Big).$$