Vector calculus MA2VC 2013–14 — Assignment 4 SOLUTIONS

(Exercise 1) We have to demonstrate

$$\iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) \mathrm{d}A = \oint_{\partial R} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}}$$

for the field and the region given; see Figure 1. We write the integrand at the left-hand side in polar coordinates as

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = 2xy \stackrel{(60)}{=} 2r^2 \cos\theta \sin\theta$$

and we compute the corresponding double integral:

$$\begin{aligned}
\iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) \mathrm{d}A &= \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) r \ \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^2 \left(\int_0^{\frac{\pi}{4}} 2r^2 \cos\theta \sin\theta \ \mathrm{d}\theta\right) r \, \mathrm{d}r \\
&= \left(\int_0^2 r^3 \, \mathrm{d}r\right)\left(\int_0^{\frac{\pi}{4}} \sin 2\theta \ \mathrm{d}\theta\right) \\
&= \frac{r^4}{4}\bigg|_0^2 \left(-\frac{1}{2}\cos 2\theta\right)\bigg|_0^{\frac{\pi}{4}} \quad = \quad \frac{16}{4}\,\frac{1}{2} \quad = \quad 2.
\end{aligned}$$

To compute the circulation at the right-hand side, we split the boundary in three parts (maintaining the anticlockwise orientation). The integration on the two straight segments (here denoted Γ_a and Γ_b as in the figure) is as follows:

$$\Gamma_a: \qquad \vec{\mathbf{a}}(t) = t\hat{\boldsymbol{\imath}}, \qquad 0 < t < 2, \qquad \frac{d\vec{\mathbf{a}}}{dt} = \hat{\boldsymbol{\imath}},$$

$$\int_{\Gamma_a} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^2 \vec{\mathbf{G}}\big(\vec{\mathbf{a}}(t)\big) \cdot \frac{d\vec{\mathbf{a}}}{dt}(t) \ \mathrm{d}t = \int_0^2 G_1(t) \ \mathrm{d}t = \int_0^2 a_1^3(t) \ \mathrm{d}t = \int_0^2 t^3 \ \mathrm{d}t = \frac{16}{4} = 4;$$

$$\Gamma_b: \qquad \vec{\mathbf{b}}(t) = \frac{1}{\sqrt{2}}(2-t)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \qquad 0 < t < 2, \qquad \frac{d\vec{\mathbf{b}}}{dt} = -\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}),$$

$$\begin{aligned}
\int_{\Gamma_b} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_0^2 \vec{\mathbf{G}}\big(\vec{\mathbf{b}}(t)\big) \cdot \frac{d\vec{\mathbf{b}}}{dt}(t) \, \mathrm{d}t \\
&= \int_0^2 \big(b_1^3(t)\hat{\boldsymbol{\imath}} + b_1^2(t) b_2(t)\hat{\boldsymbol{\jmath}}\big) \cdot \Big(-\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\Big) \, \mathrm{d}t \\
&= -\frac{1}{\sqrt{2}} \int_0^2 \big(b_1^3(t) + b_1^2(t) b_2(t)\big) \, \mathrm{d}t = -\frac{1}{\sqrt{2}} \int_0^2 2\left(\frac{1}{\sqrt{2}}(2-t)\right)^3 \mathrm{d}t = -\frac{2}{4}\frac{16}{4} = -2.
\end{aligned}$$

(Another parametrisation of Γ_b may be $\vec{\mathbf{b}}(t) = (\sqrt{2} - t)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$ for $0 < t < \sqrt{2}$.) It is immediate to see that $\vec{\mathbf{G}}(\vec{\mathbf{r}}) = x^2\vec{\mathbf{r}}$, so the field points in the radial direction (in every point $\vec{\mathbf{r}}$, $\vec{\mathbf{G}}(\vec{\mathbf{r}})$ is parallel to $\vec{\mathbf{r}}$ itself). Thus, on the arc Γ_c centred at the origin and bounding R from the right, the field $\vec{\mathbf{G}}$ is perpendicular to the arc itself. So the corresponding integral $\int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}}$ vanishes, as it depends only on the tangential component of the field. [1]

Putting everything together, we see that both left- and right-hand side of Green's theorem, in the case under examination, have value 2:

$$\oint_{\partial R} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_a} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_b} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} + \int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = 4 - 2 + 0 = 2 = \iint_R \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) \mathrm{d}A.$$

[1] If you prefer to see it in formulas, Γ_c can be parametrised by $\vec{\mathbf{c}}(t) = 2\cos t\hat{\boldsymbol{\imath}} + 2\sin t\hat{\boldsymbol{\jmath}}$ for $0 < t < \frac{\pi}{4}$, so

$$\begin{aligned}
\vec{\mathbf{G}}\big(\vec{\mathbf{c}}(t)\big) \cdot \frac{d\vec{\mathbf{c}}}{dt}(t) = \Big(c_1^3(t)\hat{\boldsymbol{\imath}} + c_1^2(t)c_2(t)\hat{\boldsymbol{\jmath}}\Big) \cdot \frac{d\vec{\mathbf{c}}}{dt}(t) &= 8(\cos^3 t\hat{\boldsymbol{\imath}} + \cos^2 t \sin t\hat{\boldsymbol{\jmath}}) \cdot (-2\sin t\hat{\boldsymbol{\imath}} + 2\cos t\hat{\boldsymbol{\jmath}}) \\
&= 16\cos^2 t(-\cos t \sin t + \sin t \cos t) = 0,
\end{aligned}$$

$$\Rightarrow \qquad \int_{\Gamma_c} \vec{\mathbf{G}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^{\frac{\pi}{4}} \vec{\mathbf{G}}\big(\vec{\mathbf{c}}(t)\big) \cdot \frac{d\vec{\mathbf{c}}}{dt}(t) \, \mathrm{d}t = \int_0^{\frac{\pi}{4}} 0 \ \mathrm{d}t = 0.$$

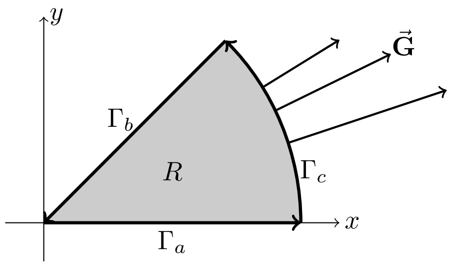

Figure 1: *The region R in exercise 1. The vector field $\vec{\mathbf{G}}$ points in the radial direction, thus it is perpendicular to Γ_c.*

(Exercise 2) Green's theorem states that we can compute the flux of $\vec{\mathbf{H}}$ through ∂D as the triple integral of its divergence on D. We first compute the divergence of the field $\vec{\mathbf{H}}$, using a vector differential identity from Proposition 1.36, the expression of the position vector $\vec{\mathbf{r}}$ and its magnitude:

$$\begin{aligned}\vec{\nabla}\cdot\vec{\mathbf{H}} = \vec{\nabla}\cdot(|\vec{\mathbf{r}}|^2\vec{\mathbf{r}}) &\overset{(27)}{=} |\vec{\mathbf{r}}|^2\vec{\nabla}\cdot\vec{\mathbf{r}} + \vec{\mathbf{r}}\cdot\vec{\nabla}(|\vec{\mathbf{r}}|^2)\\ &= |\vec{\mathbf{r}}|^2\Big(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}\Big) + \vec{\mathbf{r}}\cdot\vec{\nabla}(x^2+y^2+z^2)\\ &= 3|\vec{\mathbf{r}}|^2 + \vec{\mathbf{r}}\cdot 2\vec{\mathbf{r}} \quad = \quad 5|\vec{\mathbf{r}}|^2 \quad = \quad 5\rho^2.\end{aligned}$$

The domain D can be expressed very easily in spherical coordinates: it is the "parallelepiped" $(1,2)\times[0,\pi]\times(-\pi,\pi]$ in the $\rho\phi\theta$-space. [2] Because of this and because $\vec{\nabla}\cdot\vec{\mathbf{H}}$ depends only on one spherical coordinate (namely on ρ), the triple integral on D reduces immediately to the product of three one-dimensional integrals and the flux is:

$$\begin{aligned}\oiint_{\partial D}\vec{\mathbf{H}}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S &\overset{(88)}{=} \iiint_D \vec{\nabla}\cdot\vec{\mathbf{H}}\,\mathrm{d}V\\ &\overset{(74)}{=} \iiint_D \vec{\nabla}\cdot\vec{\mathbf{H}}\,\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta\\ &= \int_{-\pi}^{\pi}\bigg(\int_0^{\pi}\sin\phi\Big(\int_1^2\vec{\nabla}\cdot\vec{\mathbf{H}}\,\rho^2\,\mathrm{d}\rho\Big)\mathrm{d}\phi\bigg)\mathrm{d}\theta\\ &= \Big(\int_{-\pi}^{\pi}\mathrm{d}\theta\Big)\Big(\int_0^{\pi}\sin\phi\,\mathrm{d}\phi\Big)\Big(\int_1^2 5\rho^2\rho^2\,\mathrm{d}\rho\Big)\\ &= (2\pi)\,(2)\,(2^5-1^5) = 4\,\pi\,31 = 124\pi \approx 389.56.\end{aligned}$$

(We can double check the result obtained by computing directly the flux, without divergence theorem. We denote $S_O = \{|\vec{\mathbf{r}}| = 2\}$ the outer part of the boundary of D and $S_I = \{|\vec{\mathbf{r}}| = 1\}$ the inner part. Then $\vec{\mathbf{F}}\cdot\hat{\boldsymbol{n}}$ is constantly equal to 8 on S_O and to -1 on S_I (verify why). So, $\oiint_{\partial D}(\vec{\mathbf{H}}\cdot\hat{\boldsymbol{n}})\,\mathrm{d}S = 8\mathrm{Area}(S_O) - \mathrm{Area}(S_I) = 8(4\pi 2^2) - 4\pi = 124\pi$.)

[2] In the usual xyz-space the domain D is the set difference between the sphere of radius 2 and the sphere of radius 1, both centred at the origin. Thus its boundary is composed of two disconnected components. However, this fact does not affect the solution of the exercise.

(Exercise 3) *(Version 1)* We want to compute the line integral of a field along a segment lying in the line $\{y=1, z=0\}$ using its value known only on the parallel line $\{y=z=0\}$. How to establish a relation between these two lines? We introduce the square $R=(0,1)^2=\{x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath},\ 0<x,y<1\}$. We can decompose its boundary in the union of four oriented segments: $\partial R=\Gamma_S\cup\Gamma_E\cup\Gamma_N\cup\Gamma_W$, with

$$\Gamma=\Gamma_N \text{ from } \hat{\boldsymbol\imath}+\hat{\boldsymbol\jmath} \text{ to } \hat{\boldsymbol\jmath},\qquad \Gamma_W \text{ from } \hat{\boldsymbol\jmath} \text{ to } \vec{\mathbf{0}},\qquad \Gamma_S \text{ from } \vec{\mathbf{0}} \text{ to } \hat{\boldsymbol\imath},\qquad \Gamma_E \text{ from } \hat{\boldsymbol\imath} \text{ to } \hat{\boldsymbol\imath}+\hat{\boldsymbol\jmath}.$$

The idea is to use Green's theorem to relate the value of the integral along Γ with those on the other three sides of R, which can be computed using (ii) and (iii), and with a double integral computed using (i).

From Green's theorem we know that $\oint_{\partial R}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}}=\iint_R(\vec\nabla\times\vec{\mathbf{F}})_3\,\mathrm{d}A$.

Since, by property (i), $\vec{\mathbf{F}}$ is irrotational, $\vec\nabla\times\vec{\mathbf{F}}=\vec{\mathbf{0}}$ and $\oint_{\partial R}\vec{\mathbf{F}}\cdot \mathrm{d}\vec{\mathbf{r}}=\iint_R 0\,\mathrm{d}A=0$.

On the vertical segments Γ_E and Γ_W, the tangent unit vector is $\hat{\boldsymbol\tau}=\pm\hat{\boldsymbol\jmath}$ (oriented vertically), which is orthogonal to $\vec{\mathbf{F}}=F_1\hat{\boldsymbol\imath}$ because of property (ii). Thus the contributions of these sides to the circulation of $\vec{\mathbf{F}}$ vanish:

$$\int_{\Gamma_E}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\int_0^1 F_1(\hat{\boldsymbol\imath}+t\hat{\boldsymbol\jmath})\underbrace{\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\jmath}}_{=0}\,\mathrm{d}t=0,\qquad \int_{\Gamma_W}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\int_0^1 F_1\big((1-t)\hat{\boldsymbol\jmath}\big)\underbrace{\hat{\boldsymbol\imath}\cdot(-\hat{\boldsymbol\jmath})}_{=0}\,\mathrm{d}t=0.$$

Property (iii) allows to compute the line integral of $\vec{\mathbf{F}}$ along Γ_S:

$$\int_{\Gamma_S}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\int_0^1 F_1(t\hat{\boldsymbol\imath})\,\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\imath}\,\mathrm{d}t=\int_0^1 t^2\,\mathrm{d}t=\frac{1}{3}.$$

We decompose the circulation of $\vec{\mathbf{F}}$ in four terms corresponding to the four sides of S

$$0=\iint_R\underbrace{(\vec\nabla\times\vec{\mathbf{F}})_3}_{=0}\,\mathrm{d}A=\int_{\partial R}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\underbrace{\int_{\Gamma_S}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}}_{=\frac13}+\underbrace{\int_{\Gamma_E}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}}_{=0}+\int_{\Gamma_N}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}+\underbrace{\int_{\Gamma_W}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}}_{=0}.$$

We note that the upper side Γ_N coincides with the oriented path Γ in the statement of the exercise, so we can compute the desired result as

$$\int_\Gamma\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\int_{\Gamma_N}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=0-\frac13-0-0=-\frac13.$$

[3]

(Version 2, without using Green's theorem) Alternatively, we can use properties (i) and (ii) and the definition of the curl to show

$$\vec{\mathbf{0}}\overset{(i)}{=}\vec\nabla\times\vec{\mathbf{F}}\overset{(ii)}{=}\vec\nabla\times(F_1\hat{\boldsymbol\imath})\overset{(20)}{=}\frac{\partial F_1}{\partial z}\hat{\boldsymbol\jmath}-\frac{\partial F_1}{\partial y}\hat{\boldsymbol k}\qquad\Rightarrow\qquad\frac{\partial F_1}{\partial z}=\frac{\partial F_1}{\partial y}=0.$$

Thus $\vec{\mathbf{F}}=F_1\hat{\boldsymbol\imath}$ does not depend on y and z, so the field defined as $\vec{\mathbf{F}}(x,y,z)=x^2\hat{\boldsymbol\imath}$ for all $x\hat{\boldsymbol\imath}+y\hat{\boldsymbol\jmath}+z\hat{\boldsymbol k}\in\mathbb{R}^3$ is the only vector field that satisfies properties (i)–(iii). Integrating this field along Γ immediately gives the result.

[3] After defining R and its four sides as above, the entire exercise can actually be solved in just one short chain of equalities:

$$\int_\Gamma\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}=\oint_{\partial R}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}-\int_{\Gamma_W}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}-\int_{\Gamma_S}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}-\int_{\Gamma_E}\vec{\mathbf{F}}\cdot\mathrm{d}\vec{\mathbf{r}}$$

$$\overset{\text{Green's th., (ii),(iii)}}{=}\iint_R(\vec\nabla\times\vec{\mathbf{F}})_3\,\mathrm{d}A-\int_{\Gamma_W}F_1\hat{\boldsymbol\imath}\cdot(-\hat{\boldsymbol\jmath})\,\mathrm{d}s-\int_{\Gamma_S}x^2\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\imath}\,\mathrm{d}s-\int_{\Gamma_E}F_1\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\jmath}\,\mathrm{d}s\overset{(i)}{=}0-0-\frac13-0=-\frac13.$$