University of Reading
Department of Mathematics and Statistics

Useful reading: *Calculus: A Complete Course* 6^{th} Ed., R.A. Adams, Pearson Addison Wesley. *Mathematical Methods in the Physical Sciences* 3^{rd} Ed., M. Boas, Wiley.

Vector algebra

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ then the *scalar* (or *dot*) *product* $\boldsymbol{u} \cdot \boldsymbol{v}$ is the scalar (number) given by:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \begin{cases} |\boldsymbol{u}||\boldsymbol{v}|\cos\theta & \text{where } \theta \text{ is the angle between the vectors} \\ \sum_{k=1}^{n} u_k v_k = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \end{cases}$$

Geometrically, if $\boldsymbol{v}$ is a unit vector then one can interpret the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ as a measure of how much of $\boldsymbol{u}$ is projected onto $\boldsymbol{v}$ (i.e., their overlap in some sense). If $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ then $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be *orthogonal* (some authors prefer the term *perpendicular* but this concept does not generalise beyond $\mathbb{R}^3$ and as such we shall not use it).

❑ **Example.** *Let* $\boldsymbol{u} = \left(\sqrt{3} - 1, 1 + \sqrt{3}\right)^{\mathrm{T}}$ *and* $\boldsymbol{v} = \left(3, 3\right)^{\mathrm{T}}$*, as shown in figure 1, then* $\boldsymbol{u} \cdot \boldsymbol{v} = 6\sqrt{3}$*. We verify the assertions about the geometric nature of the scalar product by additionally computing:*

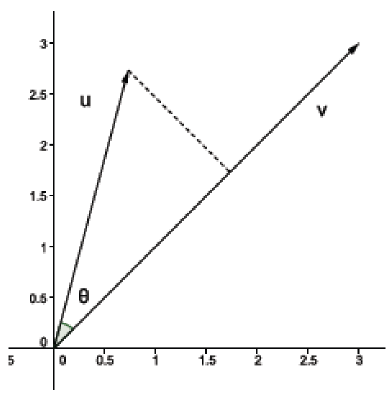

Figure 1: The scalar product of two vectors

$|\boldsymbol{u}| = 2\sqrt{2}$ *and* $|\boldsymbol{v}| = 3\sqrt{2}$*. Therefore, the angle* θ *between the vectors satisfies* $6\sqrt{3} = 12\cos\theta$ *and is consequently found to be* $\theta = \pi/6$*. Thus,* $|\boldsymbol{u}|\cos\theta$ *(this 'adjacent' component of* $\boldsymbol{u}$ *is precisely the amount of* $\boldsymbol{u}$ *projected onto* $\boldsymbol{v}$*) is* $|\boldsymbol{u}|\cos\theta = 2\sqrt{2}\cos(\pi/6) = 2\sqrt{2}\left(\sqrt{3}/2\right) = \sqrt{6}$*. To see that this is precisely the overlap of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *we rotate* $\boldsymbol{u}$ *by* $-\pi/4$ *(thereby bringing* $\boldsymbol{v}$ *in line with the horizontal axis) whereon its components become* $\left(\sqrt{6}, \sqrt{2}\right)^{\mathrm{T}}$ *which readily confirms our earlier assertion.*

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ then the *vector* (or *cross*) *product* $\boldsymbol{u} \times \boldsymbol{v}$ is the vector given by:

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \left(u_2 v_3 - u_3 v_2\right)\hat{\boldsymbol{\imath}} + \left(u_3 v_1 - u_1 v_3\right)\hat{\boldsymbol{\jmath}} + \left(u_1 v_2 - u_2 v_1\right)\hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where we formally evaluate the determinant by expanding along the first row. The vector product is constructed so as to produce a vector $\boldsymbol{u} \times \boldsymbol{v}$ that is orthogonal to both $\boldsymbol{u}$ and $\boldsymbol{v}$.

❑ **Example.** *Let* $\boldsymbol{u} = (2,-1,0)^{\mathrm{T}}$ *and* $\boldsymbol{v} = (\pi,0,1)^{\mathrm{T}}$ *then*

$$\boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & -1 & 0 \\ \pi & 0 & 1 \end{vmatrix} = \hat{\boldsymbol{\imath}}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - \hat{\boldsymbol{\jmath}}\begin{vmatrix} 2 & 0 \\ \pi & 1 \end{vmatrix} + \hat{\boldsymbol{k}}\begin{vmatrix} 2 & -1 \\ \pi & 0 \end{vmatrix} = -\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + \pi\hat{\boldsymbol{k}}\,.$$

Hence, $\boldsymbol{u}\times\boldsymbol{v} = (-1,-2,\pi)^{\mathrm{T}}$.

We check the previous assertion that $\boldsymbol{u}\times\boldsymbol{v}$ *is orthogonal to both* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *by computing*

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{u} = \begin{pmatrix} -1 \\ -2 \\ \pi \end{pmatrix}\cdot\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 & \pi \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = -2+2+0 = 0\,;$$

and

$$(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{v} = \begin{pmatrix} -1 \\ -2 \\ \pi \end{pmatrix}\cdot\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & \pi \end{pmatrix}\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix} = -\pi+0+\pi = 0\,.$$

It follows from elementary row properties of the determinant that the vector product is an antisymmetric operation; i.e., $\boldsymbol{u}\times\boldsymbol{v} = -\boldsymbol{v}\times\boldsymbol{u}$. A further corollary of this observation is that $\boldsymbol{u}\times\boldsymbol{u} = \boldsymbol{0}\ \forall\boldsymbol{u}\in\mathbb{R}^3$. Unlike the scalar product, the vector product does not generalise straightforwardly to higher dimensions (its generalisation, the wedge product, is beyond the scope of this discussion).

We close this section with some standard identities of vector algebra and a note on their proof: Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{R}^3$ then

$$\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{u}) = \boldsymbol{w}\cdot(\boldsymbol{u}\times\boldsymbol{v})$$

(note the cyclical nature) and

$$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}\,.$$

The standard strategy for establishing vector identities is to consider their components. For example, the first component, $\big[\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})\big]_1$, of $\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})$ is

$$\begin{aligned}
\big[\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w})\big]_1 &= u_2\big[\boldsymbol{v}\times\boldsymbol{w}\big]_3 - u_3\big[\boldsymbol{v}\times\boldsymbol{w}\big]_2 \\
&= u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\
&= (u_2w_2 + u_3w_3)v_1 - (u_2v_2 + u_3w_3)w_1 \\
&= (\boldsymbol{u}\cdot\boldsymbol{w} - u_1w_1)v_1 - (\boldsymbol{u}\cdot\boldsymbol{v} - u_1v_1)w_1 \\
&= (\boldsymbol{u}\cdot\boldsymbol{w})v_1 - (\boldsymbol{u}\cdot\boldsymbol{v})w_1 \\
&= \big[(\boldsymbol{u}\cdot\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{w}\big]_1
\end{aligned}$$

and to complete the proof one must establish the same result for all the other components individually.

The Levi-Civita symbol

The Levi-Civitia (or *permutation*) symbol ε_{ijk} can greatly simplify vector calculations involving the vector product, it is defined to be

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is a cyclic permutation of } 123 \\ -1 & \text{if } ijk \text{ is an anti-cyclic permutation of } 123 \\ 0 & \text{otherwise} \end{cases}$$

Consequently, for example, $\varepsilon_{ijk} = \varepsilon_{kij}$.

❑ **Example.** $\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$, $\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$ *and all other symbols vanish.*

The advantage of the Levi-Civita symbol is that one can use it to give a concise expression for the i-th component of $\boldsymbol{u} \times \boldsymbol{v}$ in the following way:

$$\big[\boldsymbol{u} \times \boldsymbol{v}\big]_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} u_j v_k \, .$$

❏ **Example.** *Recall that to compute $\big[\boldsymbol{u} \times \boldsymbol{v}\big]_1 = u_2 v_3 - u_3 v_2$ we must formally evaluate the determinant of a 3×3 'matrix' and then isolate the appropriate component. Consider*

$$\begin{aligned}
\big[\boldsymbol{u} \times \boldsymbol{v}\big]_1 &= \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} u_j v_k \\
&= \sum_{j=1}^{3} \big(\varepsilon_{1j1} u_j v_1 + \varepsilon_{1j2} u_j v_2 + \varepsilon_{1j3} u_j v_3\big) \\
&= \big(\varepsilon_{111} u_1 v_1 + \varepsilon_{121} u_2 v_1 + \varepsilon_{131} u_3 v_1\big) + \big(\varepsilon_{112} u_1 v_2 + \varepsilon_{122} u_2 v_2 + \varepsilon_{132} u_3 v_2\big) \\
&\qquad + \big(\varepsilon_{113} u_1 v_3 + \varepsilon_{123} u_2 v_3 + \varepsilon_{133} u_3 v_3\big) \\
&= \varepsilon_{132} u_3 v_2 + \varepsilon_{123} u_2 v_3 \\
&= u_2 v_3 - u_3 v_2
\end{aligned}$$

You are entitled to make the objection that this isn't a significantly easier question but it does enable for the direct manipulation of components of a vector product as we shall soon establish (especially, the curl of a vector field).

Scalar and vector fields

Throughout this section we let $\mathcal{O} \subset \mathbb{R}^n$ denote a subset of $\mathbb{R}^n$ (possibly the whole of $\mathbb{R}^n$).

Let $f : \mathcal{O} \to \mathbb{R}$ be a scalar field, i.e., a scalar valued function. For convenience, we denote the set of scalar fields by $S(\mathcal{O}, \mathbb{R})$.

❏ **Example.** *The mapping $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $(x, y, z) \mapsto xy + z$ is a scalar function; hence, $f \in S(\mathbb{R}^3, \mathbb{R})$.*

The set $S(\mathbb{R}^n, \mathbb{R})$ is easily shown to be a linear space.

❏ **Example.** *The mapping $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by $(x, y) \mapsto xy$ is a scalar function; hence, $f \in S(\mathbb{R}^2, \mathbb{R})$. The mapping $g : \mathbb{R} \to \mathbb{R}$ is defined by $(x, y) \mapsto x + y$ is a scalar function; hence, $g \in S(\mathbb{R}^2, \mathbb{R})$. As both $f, g \in S(\mathbb{R}^2, \mathbb{R})$ they may be added and their sum $(f + g) : \mathbb{R}^2 \to \mathbb{R}$, defined by $(x, y) \mapsto f(x, y) + g(x, y) = xy + x + y$, is also a scalar field.*

Let $\boldsymbol{F} : \mathcal{O} \to \mathbb{R}^n$ be a vector field, i.e., a vector valued function. For convenience, we denote the set of vector fields by $V(\mathcal{O}, \mathbb{R}^n)$. The components of a vector field are themselves scalar fields.

❏ **Example.** *The mapping $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $(x, y, z) \mapsto \big(x, 1, x + y\big)^{\mathrm{T}}$ is a vector field; hence, $\boldsymbol{F} \in V(\mathbb{R}^3, \mathbb{R}^3)$.*

The set $V(\mathcal{O}, \mathbb{R}^n)$ is easily shown to be a linear space.

❏ **Example.** *The mapping $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $(x, y) \mapsto \big(x, y\big)^{\mathrm{T}}$ is a vector field; that is, $\boldsymbol{F} \in V(\mathbb{R}^2, \mathbb{R}^2)$. The mapping $\boldsymbol{G} : \mathbb{R} \to \mathbb{R}$ is defined by $(x, y) \mapsto \big(x, x + y\big)^{\mathrm{T}}$ is a vector field; that is, $\boldsymbol{G} \in V(\mathbb{R}^2, \mathbb{R}^2)$. As both $\boldsymbol{F}, \boldsymbol{G} \in V(\mathbb{R}^2, \mathbb{R}^2)$ they may be added and their sum $(\boldsymbol{F} + \boldsymbol{G}) : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $(x, y) \mapsto \boldsymbol{F}(x, y) + \boldsymbol{G}(x, y) = \big(2x, x + 2y\big)^{\mathrm{T}}$, is also a vector field.*

grad, div and curl

Throughout this section we let $\mathcal{O} \subset \mathbb{R}^n$ denote a subset of $\mathbb{R}^n$ (possibly the whole of $\mathbb{R}^n$).

The *gradient operator* $\boldsymbol{\nabla} : S(\mathcal{O}, \mathbb{R}) \to V(\mathcal{O}, \mathbb{R}^n)$ is defined by

$$f \mapsto \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)^{\mathrm{T}}$$

for all sufficiently differentiable $f \in S(\mathcal{O}, \mathbb{R})$.

❏ **Example.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x, y) \mapsto xy + \cos x$ then $\boldsymbol{\nabla} f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is given by*

$$(x, y) \mapsto (y - \sin x, x)^{\mathrm{T}} \ .$$

Typically, we write $\boldsymbol{\nabla} f(x, y) = (y - \sin x, x)^{\mathrm{T}}$ or $\boldsymbol{\nabla} f(x, y) = (y - \sin x)\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$.

Let $\sigma : \mathbb{R}^3 \to \mathbb{R}$ then $\sigma(x, y, z) = 0$ defines a two-dimensional surface in $\mathbb{R}^3$ and the vector $\boldsymbol{\nabla}\sigma$ is everywhere normal (orthogonal) to the surface defined by $\sigma(x, y, z) = 0$.

❏ **Example.** *Let $\sigma : \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x, y, z) \mapsto x + y - 3z + 5$ then $\sigma(x, y, z) = 0$ defines a two-dimensional surface in $\mathbb{R}^3$ (indeed, $x + y - 3z + 5 = 0$ is an equation of a plane whose normal vector is easily inspected to be $(1, 1, -3)^{\mathrm{T}}$). We compute*

$$\boldsymbol{\nabla}\sigma = \left(\frac{\partial}{\partial x}(x + y - 3z + 5), \frac{\partial}{\partial y}(x + y - 3z + 5), \frac{\partial}{\partial z}(x + y - 3z + 5) \right)^{\mathrm{T}} = (1, 1, -3)^{\mathrm{T}}$$

as expected.

❏ **Example.** *Let $\sigma : \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x, y, z) \mapsto x^2 + y^2 + z^2 - 9$ then $\sigma(x, y, z) = 0$ defines a two-dimensional surface in $\mathbb{R}^3$ (indeed, $x^2 + y^2 + z^2 - 9 = 0$ defines the surface of the sphere of radius 3 centred at the origin). We compute $\boldsymbol{\nabla}\sigma = (2x, 2y, 2z)^{\mathrm{T}}$ which is certainly an outward normal vector (consider the various octants for confirmation).*

Let $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field whose components $\boldsymbol{F} = \big(F_1, \dots, F_n\big)^{\mathrm{T}}$ are each sufficiently differentiable scalar fields. The *divergence of the vector field* $\boldsymbol{F}$, denoted $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, is the scalar field given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k} \, .$$

(One can think of the *divergence operator* $\boldsymbol{\nabla}\cdot : V(\mathcal{O}, \mathbb{R}) \to S(\mathcal{O}, \mathbb{R})$ should one wish to).

❏ **Example.** *Consider the $\mathbb{R}^3$ vector field $\boldsymbol{F}$ given by $(x, y, z) \mapsto (x^2, yz \sin x, 1)^{\mathrm{T}}$ then $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ is given by*

$$(x, y, z) \mapsto \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(yz \sin x) + \frac{\partial}{\partial z}(1) = 2x + z \sin x \, ;$$

alternatively written $\boldsymbol{\nabla} \cdot \boldsymbol{F}(x, y, z) = 2x + z \sin x$.

Let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field as before. The *curl of the vector field* $\boldsymbol{F}$, denoted $\boldsymbol{\nabla} \times \boldsymbol{F}$, is the $\mathbb{R}^3$ vector field given by

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\boldsymbol{k}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \, .$$

Note that again when we write the curl of a field in terms of a determinant that this is pure formalism. (Again, one can think of the *curl operator* $\boldsymbol{\nabla}\times : V(\mathcal{O}, \mathbb{R}) \to V(\mathcal{O}, \mathbb{R})$ should one wish to).

It is at this point that we have the opportunity to adopt a more powerful notation. Consider the arbitrary $\mathbb{R}^3$ vector $\boldsymbol{x} = (x, y, z)^{\mathrm{T}}$; rather than using the usual Cartesian labels we instead write $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}}$. The advantage of this notation is that one can now write i-th component ∇_i of the gradient operator $\boldsymbol{\nabla}$ is $\nabla_i = \partial/\partial x_i$.

❑ **Example.** *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be defined by* $(x_1, x_2, x_3) \mapsto x_1 + x_1 x_2$ *then*

$$\big[\boldsymbol{\nabla} f\big]_1 = \nabla_1 f = 1 + x_2\,, \quad \big[\boldsymbol{\nabla} f\big]_2 = \nabla_2 f = x_1 \quad \textit{and} \quad \big[\boldsymbol{\nabla} f\big]_3 = \nabla_3 f = 0\,.$$

A consequence of this new notation is the following expression for the curl $\boldsymbol{\nabla} \times \boldsymbol{F}$ of the vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ with components $\big(F_1, F_2, F_3\big)^{\mathrm{T}}$ where $F_1, F_2, F_3 \in S(\mathbb{R}^3, \mathbb{R})$:

$$\big[\boldsymbol{\nabla} \times \boldsymbol{F}\big]_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_j F_k\,.$$

Using this notation one can begin to establish those vector calculus identities involving the curl. For example, let $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field (as before) then

$$\nabla_i \big[\boldsymbol{\nabla} \times \boldsymbol{F}\big]_i = \nabla_i \left(\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_j F_k \right) = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_i \nabla_j F_k\,.$$

Summing the term $\nabla_i \big[\boldsymbol{\nabla} \times \boldsymbol{F}\big]_i$ over all admissible i one obtains

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} \nabla_i \nabla_j F_k \\
&= \varepsilon_{123} \nabla_1 \nabla_2 F_3 + \varepsilon_{312} \nabla_3 \nabla_1 F_2 + \varepsilon_{231} \nabla_2 \nabla_3 F_1 \\
&\qquad + \varepsilon_{321} \nabla_3 \nabla_2 F_1 + \varepsilon_{132} \nabla_1 \nabla_3 F_2 + \varepsilon_{231} \nabla_2 \nabla_3 F_1 \\
&= \big(\nabla_1 \nabla_2 - \nabla_2 \nabla_1\big) F_3 + \big(\nabla_3 \nabla_1 - \nabla_1 \nabla_3\big) F_2 \\
&\qquad + \big(\nabla_2 \nabla_3 - \nabla_3 \nabla_2\big) F_1 \\
&= 0
\end{aligned}$$

as partial derivatives commute (i.e., $\nabla_1 \nabla_2 F_3 = \nabla_2 \nabla_1 F_3$, etc.) for sufficiently well behaved functions (Clairaut's theorem). Thus we learn the famous vector calculus identity that "div-curl" vanishes.

We close this section with the remark that vector calculus has many important applications in science outside of its interest in mathematics.

❑ **Example** (Maxwell's equations)**.** *An electric field* $\boldsymbol{E}$ *with associated magnetic field* $\boldsymbol{B}$ *satisfy the* Maxwell equations*:*

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho\,, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0\,, \quad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \textit{and} \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t}$$

where ρ *is the charge density,* $\boldsymbol{j}$ *is the current and the auxiliary vectors* $\boldsymbol{D}$ *and* $\boldsymbol{H}$ *(for a field propagating in an isotropic and homogeneous medium) satisfy* $\boldsymbol{E} \propto \boldsymbol{D}$ *and* $\boldsymbol{B} \propto \boldsymbol{H}$*. Any electromagnetic field must obey all of Maxwell's equations.*

Standard identities involving grad, div and curl

In this brief section we cover the basic identities of vector calculus beginning with a familiar object from the study of differential equations. Recall the *Laplacian* $\triangle$ which typically features in the Laplace, Poisson or heat equation; in two-dimensions $\triangle$ is given by $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ or in n-dimensions

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\,.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable scalar field then

$$\big(\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\big) f = \boldsymbol{\nabla} \cdot \big(\boldsymbol{\nabla} f\big) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_1} + \ldots \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_n} = \triangle f\,.$$

Hence, we obtain the standard interpretation of the Laplacian as "div-grad": $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \triangle$.

We may prove other vector calculus identities by adopting this strategy. For example, let $\boldsymbol{F},\boldsymbol{G} \in V(\mathbb{R}^3,\mathbb{R}^3)$ be suitably differentiable vector fields and consider

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\big(\boldsymbol{F}\times\boldsymbol{G}\big) &= \sum_{i=1}^{3}\nabla_i\big(\boldsymbol{F}\times\boldsymbol{G}\big)_i \\
&= \sum_{i=1}^{3}\nabla_i\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}F_jG_k \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon_{ijk}\big((\nabla_iF_j)G_k + F_j\nabla_iG_k\big) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\big(\varepsilon_{kij}((\nabla_iF_j)G_k - \varepsilon_{jik}F_j\nabla_iG_k\big) \\
&= \sum_{k=1}^{3}\big(\boldsymbol{\nabla}\times\boldsymbol{F}\big)_kG_k - \sum_{j=1}^{3}F_j\big(\boldsymbol{\nabla}\times\boldsymbol{G}\big)_j \\
\Leftrightarrow\quad \boldsymbol{\nabla}\cdot\big(\boldsymbol{F}\times\boldsymbol{G}\big) &= \big(\boldsymbol{\nabla}\times\boldsymbol{F}\big)\cdot\boldsymbol{G} - \boldsymbol{F}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{G}\big)\,.
\end{aligned}$$

By continuing in this manner one can establish the following standard vector calculus identities:

$$\begin{aligned}
\boldsymbol{\nabla}\big(fg\big) &= f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f & \boldsymbol{\nabla}\cdot\big(\boldsymbol{F}\times\boldsymbol{G}\big) &= \big(\boldsymbol{\nabla}\times\boldsymbol{F}\big)\cdot\boldsymbol{G} - \boldsymbol{F}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{G}\big) \\
\boldsymbol{\nabla}\big(f\boldsymbol{F}\big) &= \big(\boldsymbol{\nabla}f\big)\cdot\boldsymbol{F} + f\big(\boldsymbol{\nabla}\cdot\boldsymbol{F}\big) & \boldsymbol{\nabla}\times\big(\boldsymbol{F}\times\boldsymbol{G}\big) &= \boldsymbol{F}\big(\boldsymbol{\nabla}\cdot\boldsymbol{G}\big) - \boldsymbol{G}\big(\boldsymbol{\nabla}\cdot\boldsymbol{F}\big) \\
& & &\quad - \big(\boldsymbol{F}\cdot\boldsymbol{\nabla}\big)\boldsymbol{G} + \big(\boldsymbol{G}\cdot\boldsymbol{\nabla}\big)\boldsymbol{F} \\
\boldsymbol{\nabla}\times\big(f\boldsymbol{F}\big) &= \big(\boldsymbol{\nabla}f\big)\times\boldsymbol{F} + f\big(\boldsymbol{\nabla}\times\boldsymbol{F}\big) & \boldsymbol{\nabla}\big(\boldsymbol{F}\cdot\boldsymbol{G}\big) &= \boldsymbol{F}\times\big(\boldsymbol{\nabla}\times\boldsymbol{G}\big) + \boldsymbol{G}\times\big(\boldsymbol{\nabla}\times\boldsymbol{F}\big) \\
& & &\quad + \big(\boldsymbol{F}\cdot\boldsymbol{\nabla}\big)\boldsymbol{G} + \big(\boldsymbol{G}\cdot\boldsymbol{\nabla}\big)\boldsymbol{F} \\
\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}f\big) &= \boldsymbol{0} & \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{F}\big) &= 0 \\
\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f &= \triangle f & \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\boldsymbol{F}\big) &= \boldsymbol{\nabla}\big(\boldsymbol{\nabla}\cdot\boldsymbol{F}\big) - \triangle\boldsymbol{F}
\end{aligned}$$

for any scalar fields f,g and vector fields $\boldsymbol{F},\boldsymbol{G}$.

Line integrals

Let $a<b$, $f:\mathbb{R}^n\to\mathbb{R}$ be a scalar field and consider the curve $\boldsymbol{\gamma}:\mathbb{R}\to\mathbb{R}^n$. For convenience we define the set $\Gamma\subset\mathbb{R}^n$ by $\Gamma=\{\boldsymbol{\gamma}(t)\mid t\in[a,b]\}$. The line integral

$$\int_\Gamma f\,\mathrm{d}s = \int_a^b f(\boldsymbol{\gamma}(t))|\boldsymbol{\gamma}'(t)|\,\mathrm{d}t$$

where $\mathrm{d}s$ is an infinitesimal piece of the curve $\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}(a)=\boldsymbol{\gamma}(b)$ then we say the line integral is a *contour integral* and denote thus

$$\oint_\Gamma f\,\mathrm{d}s\,.$$

❏ **Example.** *Consider the points $(0,0)$ and $(1,1)$ and the connecting path $\boldsymbol{\gamma}:\mathbb{R}\to\mathbb{R}^2$ defined by $t\mapsto(t,t)$ $\forall t\in[0,1]$. Then $\boldsymbol{\gamma}'(t)=(1,1)^{\mathrm{T}}$, thus $|\boldsymbol{\gamma}'(t)|=\sqrt{2}$; hence,*

$$\int_\Gamma\big(x^2+y\big)\,\mathrm{d}s = \int_0^1\big(t^2+t\big)\sqrt{2}\,\mathrm{d}t = \frac{5}{3\sqrt{2}}\,.$$

❏ **Example.** *Consider the points $(0,0)$ and $(1,1)$ and the connecting path $\boldsymbol{\gamma}:\mathbb{R}\to\mathbb{R}^2$ defined by $t\mapsto(\sin(\pi t/2),\sin(\pi t/2))$ $\forall t\in[0,1]$. Then $\boldsymbol{\gamma}'(t)=(\pi/2)\cos(\pi t/2)(1,1)^{\mathrm{T}}$, thus $|\boldsymbol{\gamma}'(t)|=(\pi/2)|\cos(\pi t/2)|$; hence,*

$$\int_\Gamma\big(x^2+y\big)\,\mathrm{d}s = \int_0^1\big(\sin^2(\pi t/2)+\sin(\pi t/2)\big)\frac{\pi}{\sqrt{2}}\cos(\pi t/2)\,\mathrm{d}t = \frac{5}{3\sqrt{2}}\,.$$

Let $a < b$, $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded vector field and consider the curve $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^n$. For convenience we define the set $\Gamma \subset \mathbb{R}^n$ by $\Gamma = \{\boldsymbol{\gamma}(t) \mid t \in [a, b]\}$. The line integral

$$\int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_a^b \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) \, \mathrm{d}t$$

where $\mathrm{d}\boldsymbol{r} = \big(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z\big)^{\mathrm{T}}$. If $\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$ then we say the line integral is a *contour integral* and denote it thus

$$\oint_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \, .$$

Contrary to intuition it is not *a priori* true that $\oint_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$; in fact, it is generally not true.

❏ **Example.** *Consider the points* $(0, 0)$ *and* $(1, 1)$ *and the connecting path* $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ *defined by* $t \mapsto (t, t)$ $\forall t \in [0, 1]$*. Then* $\boldsymbol{\gamma}'(t) = (1, 1)^{\mathrm{T}}$*; hence,*

$$\int_\Gamma \big(y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}\big) \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{pmatrix} t \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{d}t = \int_0^1 0 \, \mathrm{d}t = 0 \, .$$

❏ **Example.** *Consider the points* $(0, 0)$ *and* $(1, 1)$ *and the connecting path* $\boldsymbol{\gamma} : \mathbb{R} \to \mathbb{R}^2$ *defined by* $t \mapsto (t, \sin(\pi t/2))$ $\forall t \in [0, 1]$*. Then* $\boldsymbol{\gamma}'(t) = (1, (\pi/2)\cos(\pi t/2))^{\mathrm{T}}$*; hence,*

$$\begin{aligned}
\int_\Gamma \big(y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}\big) \cdot \mathrm{d}\boldsymbol{r} &= \int_0^1 \begin{pmatrix} \sin(\pi t/2) \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ (\pi/2)\cos(\pi t/2) \end{pmatrix} \mathrm{d}t \\
&= \int_0^1 \Big(\sin(\pi t/2) - \frac{\pi}{2} t \cos(\pi t/2)\Big) \, \mathrm{d}t \\
&= \frac{4 - \pi}{\pi} \, .
\end{aligned}$$

Note that the results are different depending on the path taken between the end points of the curve.

Conservative vector fields and scalar potentials

A vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *conservative* if there exists a $f : \mathbb{R}^n \to \mathbb{R}$ such that $\boldsymbol{F} = \boldsymbol{\nabla} f$. The function f is referred to as a *scalar potential*. (Some authors insist on $\boldsymbol{F} = -\boldsymbol{\nabla} f$; this alternative convention choice has considerable advantages in the theory of classical mechanics and electromagnetism).

❏ **Example.** *The vector field* $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $(x, y) \mapsto \big(x + y, x\big)^{\mathrm{T}}$ *is conservative; indeed, it arises from the scalar potential* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $(x, y) \mapsto xy + x^2/2$*.*

❏ **Example.** *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be given by* $(x_1, x_2, x_3) \mapsto x_1 x_2 x_3$ *then the vector field* $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *defined by*

$$(x_1, x_2, x_3) \mapsto \big(\nabla_1 f, \nabla_2 f, \nabla_3 f\big)^{\mathrm{T}} = \big(x_2 x_3, x_1 x_3, x_1 x_2\big)^{\mathrm{T}}$$

is conservative.

❏ **Example.** *Consider the vector field* $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $(x, y) \mapsto \big(y + 1, x\big)^{\mathrm{T}}$ *which is easily shown to be conservative by solving*

$$\frac{\partial f}{\partial x} = y + 1 \quad \ldots ① \qquad \frac{\partial f}{\partial y} = x \quad \ldots ② \, .$$

Integration (w.r.t. x*) of equation ① implies that* $f(x, y) = x(y+1) + \varphi(y)$ *for some arbitrary function* φ*. This proposed general solution must also satisfy equation ②, i.e.,* $x + \varphi'(y) = x \Leftrightarrow \varphi'(y) = 0$*; the solution* φ *to this* ODE *is* $y \mapsto c$ *where* c *is an arbitrary constant of integration. Thus, the general scalar potential solution* f *is given by* $(x, y) \mapsto x(y + 1) + c$*, however, in practice it is typical to drop the constant as* f *only appears under a divergence or as a difference term when this cannot influence the answer. Therefore,* $f(x, y) = x(y + 1)$*. Hence, the vector field is conservative.*

Not all vector fields are conservative. To prove a given vector field is not conservative one must establish that there does not exist a scalar potential, typically one use a *reductio ad absurdum* argument.

❏ **Example.** *The vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto \left(y, -x\right)^{\mathrm{T}}$ is not conservative; indeed, suppose it were and that there exist a scalar potential $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying*

$$\frac{\partial f}{\partial x} = y \quad \ldots① \qquad \frac{\partial f}{\partial y} = -x \quad \ldots②\,.$$

Integration (w.r.t. x) of equation ① implies that $f(x, y) = xy + \varphi(y)$ for some arbitrary function φ. This proposed general solution must also satisfy equation ②, i.e., $x + \varphi'(y) = -x$ but no such function of a single variable can exist (our desired contradiction). Hence, the vector field is not conservative.

An $\mathbb{R}^3$ conservative vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$. This is a straightforward corollary of $\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} f\right) = \boldsymbol{0}$. A vector field of vanishing curl is said to be *irrotational*.

The fundamental theorem of vector calculus

Let $\Gamma \subset \mathbb{R}^n$ define a closed path between the points p and q then the *fundamental theorem of vector calculus* states that: if $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a conservative vector field then

$$\int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = f\Big|_p^q = f(q) - f(p)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a scalar potential associated to $\boldsymbol{F}$ (i.e., $\boldsymbol{F} = \boldsymbol{\nabla} f$). We can make the connection with the usual one variable situation by writing the result exclusively in terms of the scalar potential whereon it becomes $\int_\Gamma \left(\boldsymbol{\nabla} f\right) \cdot \mathrm{d}\boldsymbol{r} = f\big|_p^q$ which is pleasingly reminiscent of the standard fundamental theorem of calculus.

❏ **Example.** *Consider the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto \left(y + 1, x\right)^{\mathrm{T}}$ which we have previously established is conservative with associated scalar potential $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x(y + 1)$ and the points $p = (0, 0)$ and $q = (1, 1)$. Then*

$$\int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = f(q) - f(p) = x(y + 1)\Big|_{(0,0)}^{(1,1)} = 2$$

independently of the specific path Γ joining the endpoints. To further reinforce this remarkable fact we specify a particular path Γ by $\Gamma = \{(t, t) \in \mathbb{R}^2 \mid t \in [0, 1]\}$ then

$$\int_\Gamma \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{pmatrix} t + 1 \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{d}t = \int_0^1 \left(2t + 1\right) \mathrm{d}t = 2$$

as expected.

❏ **Example.** *We have previously established that the vector field $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto \left(y, -x\right)^{\mathrm{T}}$ is not conservative. The points $p = (0, 0)$ and $q = (1, 1)$ can be connected by a variety of paths, for definiteness we consider the paths*

$$\begin{aligned} \Gamma_1 &= \{(t, t) \in \mathbb{R}^2 \mid t \in [0, 1]\}\,; \\ \Gamma_2 &= \{(t, t^2) \in \mathbb{R}^2 \mid t \in [0, 1]\}\,; \\ \Gamma_3 &= \{(0, s) \in \mathbb{R}^2 \mid s \in [0, 1]\} \cup \{(t, 1) \in \mathbb{R}^2 \mid t \in [0, 1]\}\,. \end{aligned}$$

Then

$$\int_{\Gamma_1} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{pmatrix} t \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mathrm{d}t = \int_0^1 0 \,\mathrm{d}t = 0\,.$$

Whereas,

$$\int_{\Gamma_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{pmatrix} t^2 \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} \mathrm{d}t = \int_0^1 \left(-t^2\right) \mathrm{d}t = -\frac{1}{3}\,,$$

and, finally,

$$\int_{\Gamma_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_0^1 \begin{pmatrix} s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}s + \int_0^1 \begin{pmatrix} 1 \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t = \int_0^1 0\,\mathrm{d}s + \int_0^1 1\,\mathrm{d}t = 1\,.$$

Hence,

$$\int_{\Gamma_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \neq \int_{\Gamma_2} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \neq \int_{\Gamma_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \quad \textit{and} \quad \int_{\Gamma_1} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} \neq \int_{\Gamma_3} \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r}$$

despite the fact that Γ_1, Γ_2 *and* Γ_3 *all have the same endpoints.*

Path independence of line integral quantities is not just a pleasing mathematical fact, it is of central importance of the field of classical physics and thermodynamics where it has profound consequences. For example, the work done by to resist a force $\boldsymbol{F}$ along the path Γ is given by $\int_\Gamma \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$; hence, if the force is not conservative one may have to work harder along certain paths between the endpoints of the motion.

Multiple integration

Let $a < b$, $g : [a, b] \to \mathbb{R}$ be a bounded function and consider the expression

$$\int_a^b g(x)\,\mathrm{d}x$$

which we typically interpret as the (signed) area between the curves $x = a$, $x = b$, $y = 0$ and $y = g(x)$. This interpretation rests on the fact that $g(x)\mathrm{d}x$ can be identified as the area of the rectangle $[x, x + \mathrm{d}x] \times [0, g(x)]$ which one sums (continuously) over all admissible $x \in [a, b]$. We wish to extend this notion and consider the integral of a function of two (and later more) variables.

Suppose $\mathcal{R} \subset \mathbb{R}^2$ is a bounded region and $f : \mathcal{R} \to \mathbb{R}$ is a bounded function of two variables. We wish to give meaning to expressions of the form

$$\int_{\mathcal{R}} f(x, y)\,\mathrm{d}A$$

where the region of integration is $\mathcal{R}$ and the area element is $\mathrm{d}A$. If we consider an infinitesimal element of area $\mathrm{d}A$ in Cartesian coordinates of the rectangular region with vertices (x, y), $(x, +\mathrm{d}x, y)$, $(x+\mathrm{d}x, y)$ and $(x + \mathrm{d}x, y + \mathrm{d}y)$ as shown in figure 2. This indicates that, in Cartesian coordinates, $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ and that we should be able to evaluate the area integral by a double integral. The question remains as to the limits of integration in the variables:

$$\int_{\mathcal{R}} f(x,y)\mathrm{d}A = \int_{??}^{?} \left(\int_{????}^{???} f(x,y)\mathrm{d}x\right) \mathrm{d}y \quad \text{or} \quad \int_{\mathcal{R}} f(x,y)\mathrm{d}A = \int_{\overline{??}}^{\overline{?}} \left(\int_{\overline{????}}^{\overline{???}} f(x,y)\mathrm{d}y\right) \mathrm{d}x$$

(as, at least intuitively, the area $\mathrm{d}x\,\mathrm{d}y$ is the same as $\mathrm{d}y\,\mathrm{d}x$).

The answer is provided by considering the boundary $\partial\mathcal{R}$ of $\mathcal{R}$. The boundary $\partial\mathcal{R}$ is a curve in $\mathbb{R}^2$, that is a one dimensional object. Let us denote the equation of the curve by $\varphi(x, y) = 0$ which one may solve for either variable (we are at liberty to consider either as the independent one). Consider figure 3 which indicates that we may evaluate these double integrals using the following limit prescription:

$$\int_{\mathcal{R}} f(x,y)\mathrm{d}A = \int_{\min y}^{\max y} \left(\int_{\min x(y)}^{\max x(y)} f(x,y)\mathrm{d}x\right) \mathrm{d}y$$

or

$$\int_{\mathcal{R}} f(x,y)\mathrm{d}A = \int_{\min x}^{\max x} \left(\int_{\min y(x)}^{\max y(x)} f(x,y)\mathrm{d}y\right) \mathrm{d}x$$

as desired. In the examples which follow the reader is strongly advised to verify the asserted limits by means of a suitable sketch (in the form of figure 3).

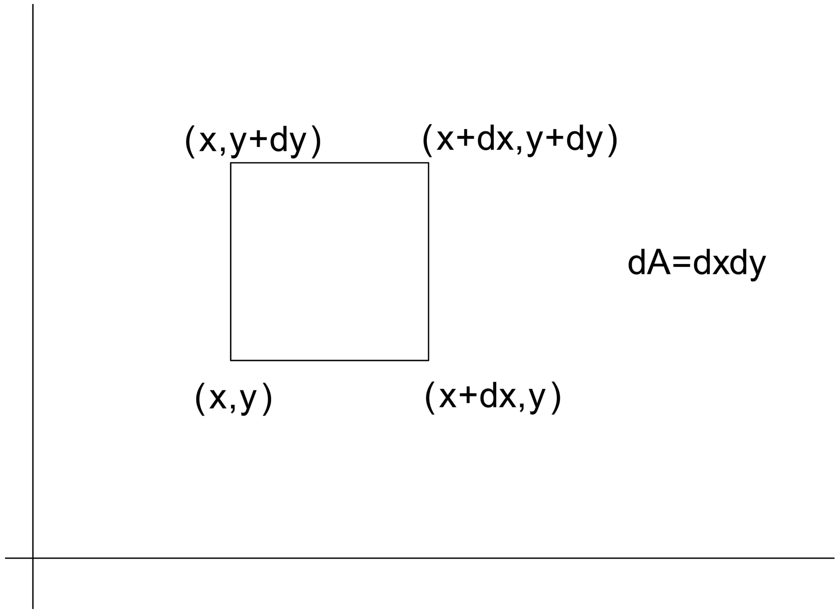

Figure 2: The area element $\mathrm{d}A$ in Cartesian coordinates

❑ **Example.** *Let $\mathcal{R} = [0,1] \times [1,2]$ and $f : \mathcal{R} \to \mathbb{R}$ be defined by $(x,y) \mapsto x^2 + x\cos\pi y$ then*

$$\int_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \int_1^2 \left(\int_0^1 (x^2 + x\cos\pi y)\mathrm{d}x \right) \mathrm{d}y = \int_1^2 \left(\frac{1}{3} + \frac{1}{2}\cos\pi y \right) \mathrm{d}y = \frac{1}{3}\,.$$

❑ **Example.** *Let $\mathcal{R}$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$ and $f : \mathcal{R} \to \mathbb{R}$ be given by $(x,y) \mapsto \sinh y$ then*

$$\int_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \int_0^3 \left(\int_{y/3}^1 \sinh y\,\mathrm{d}x \right)\,\mathrm{d}y = \int_0^3 \sinh y \left(1 - \frac{y}{3}\right)\,\mathrm{d}y = -1 + \frac{1}{3}\sinh 3\,.$$

❑ **Example.** *Let $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid x + y^2 \le 1\}$ and $f : \mathcal{R} \to \mathbb{R}$ be defined by $(x,y) \mapsto xy$ then*

$$\int_{\mathcal{R}} f(x,y)\,\mathrm{d}A = \int_{-1}^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y^2)\,\mathrm{d}x \right) \mathrm{d}y = \int_{-1}^1 2y^2\sqrt{1-y^2}\,\mathrm{d}y = \frac{\pi}{4}\,.$$

We remark that this integral is remarkably symmetric in the following sense. Suppose we perform the change of variables $y \mapsto \tilde{y} = -y$ then

$$\begin{aligned} \int_{-1}^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y^2)\,\mathrm{d}x \right) \mathrm{d}y &= \int_1^{-1} \left(\int_{-\sqrt{1-\tilde{y}^2}}^{\sqrt{1-\tilde{y}^2}} (x + \tilde{y}^2)\,\mathrm{d}x \right) (-\,\mathrm{d}\tilde{y}) \\ &= \int_{-1}^1 \left(\int_{-\sqrt{1-\tilde{y}^2}}^{\sqrt{1-\tilde{y}^2}} (x + \tilde{y}^2)\,\mathrm{d}x \right) \mathrm{d}\tilde{y}\,. \end{aligned}$$

Therefore,

$$\int_{-1}^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y^2)\,\mathrm{d}x \right) \mathrm{d}y = 2\int_0^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x + y^2)\,\mathrm{d}x \right) \mathrm{d}y$$

which may be equivalently expressed

$$\int_{\mathcal{R}} (x^2 + y^2)\,\mathrm{d}A = 2\int_{\mathcal{D}} (x^2 + y^2)\,\mathrm{d}A$$

where $\mathcal{D}$ is the upper unit semi-circle centred at the origin. Thus, $\int_{\mathcal{D}} (x^2 + y^2)\,\mathrm{d}A = \pi/8$.

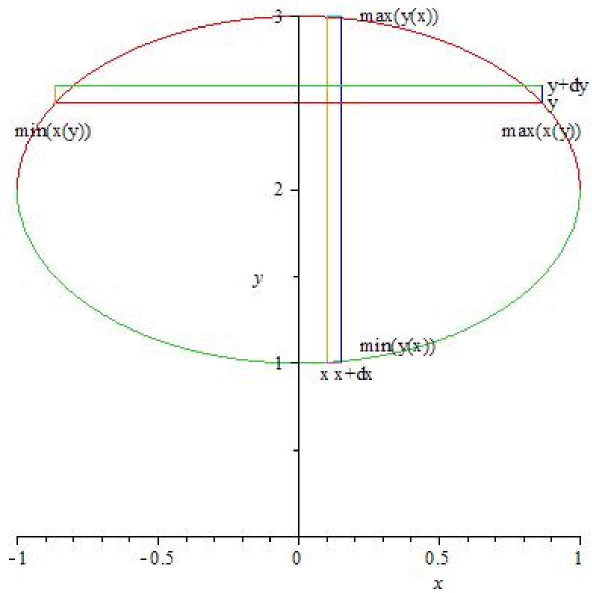

Figure 3: The limits of a double integral

Polar coordinates

Some situations admit particular symmetry which may be exploited to enable easier computation. In this section we briefly consider planar circular symmetry where the polar coordinates $(r,\theta)\in[0,\infty)\times[0,2\pi)$ are related to the Cartesian coordinates $(x,y)\in\mathbb{R}^2$ through $x=r\cos\theta$ and $y=r\sin\theta$. By considering an infinitesimal area $\mathrm{d}A$ element in planar polar coordinates one can show that $\mathrm{d}A=r\mathrm{d}r\,\mathrm{d}\theta$; see figure 4.

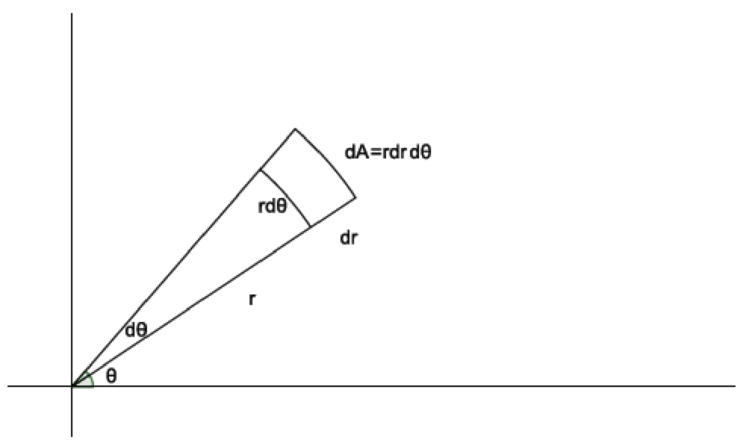

Figure 4: Area element in polar coordinates

❑ **Example.** *Let $\mathcal{R}$ be the upper half of the unit circle centred at the origin in $\mathbb{R}^2$ and $f:\mathcal{R}\to\mathbb{R}$ be given by $(x,y)\mapsto x+y^2$ then*

$$\int_{\mathcal{R}} f(x,y)\,\mathrm{d}A=\int_0^1\left(\int_0^\pi\left(r\cos\theta+r^2\sin^2\theta\right)\mathrm{d}\theta\right)r\mathrm{d}r=\int_0^1\left(\frac{\pi}{2}r^2\right)r\mathrm{d}r=\frac{\pi}{8}$$

as expected.

Multiple integrals and the Jacobian determinant

One may extend the idea of double integrals to higher dimensional settings. For the purposes of this discussion we shall restrict ourselves to a brief remark on higher integrals of the form

$$\int_\Omega f(x_1,\dots,x_n)\,\mathrm{d}V$$

where $\Omega \subset \mathbb{R}^n$, $f:\Omega\to\mathbb{R}$ is a bounded function and the volume element $\mathrm{d}V$ (in Cartesian coordinates) is given by $\mathrm{d}V = \prod_{k=1}^n \mathrm{d}x_k = \mathrm{d}x_1\dots\mathrm{d}x_n$.

❏ **Example.** *Let $a<b$ and $\Omega=[a,b]\times\cdots\times[a,b]$ (n-times) then*

$$\int_\Omega \mathrm{d}V = \int_a^b \left(\dots\left(\int_a^b \mathrm{d}x_1\right)\dots\right)\mathrm{d}x_n = \big(b-a\big)^n$$

as one would expect for the 'volume' of the n-cube of side length $b-a$.

❏ **Example.** *Let $h,\ell>0$, $\Omega=\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le\ell^2$ and $z\in[0,h]\}=\mathcal{R}\times[0,h]$ where $\mathcal{R}=\{(x,y)\in\mathbb{R}^2\mid x^2+y^2\le\ell^2\}$ and consider $\int_\Omega \mathrm{d}V$ which we expect, on geometric grounds, to be equal to $\pi\ell^2 h$. Indeed, one may verify this by writing*

$$\int_\Omega \mathrm{d}V = \int_{\mathcal{R}\times[0,h]}\mathrm{d}V = \int_{\mathcal{R}}\mathrm{d}A\int_0^h \mathrm{d}z = h\int_{\mathcal{R}}\mathrm{d}A = h\int_0^\ell\left(\int_0^{2\pi}\mathrm{d}\theta\right)r\mathrm{d}r = h\int_0^\ell(2\pi)r\mathrm{d}r = \pi\ell^2 h\,.$$

We turn our attention briefly to the subject of the change of variables in multiple integration. Suppose we wish to change from Cartesian variables $(x_1,\dots,x_n)$ to another set of variables $(t_1,\dots,t_n)$ then one can show that

$$\mathrm{d}x_1\dots\mathrm{d}x_n = \left|\frac{\partial(x_1,\dots,x_n)}{\partial(t_1,\dots,t_n)}\right|\mathrm{d}t_1\dots\mathrm{d}t_n \quad\text{where}\quad \frac{\partial(x_1,\dots,x_n)}{\partial(t_1,\dots,t_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_n}\end{vmatrix}$$

is the *Jacobian determinant* of the change of variables transform.

❏ **Example.** *Consider the change from Cartesian to planar polar coordinates via $x=r\cos\theta$ and $y=r\sin\theta$ then*

$$\frac{\partial(x_1,\dots,x_n)}{\partial(t_1,\dots,t_n)} = \begin{vmatrix}\partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r\,.$$

Hence,

$$\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$$

as before.

❏ **Example** (Spherical polar coordinates)**.** *Let $\ell>0$, $\Omega=\{(x,y,z)\in\mathbb{R}^3\mid x^2+y^2+z^2\le\ell\}$ and define $f:\Omega\to\mathbb{R}$ by $(x,y,z)\to x^2+y^2$ then $\int_\Omega f(x,y,z)\,\mathrm{d}V$ may be evaluated by the following change of variables*

$$\left.\begin{array}{l} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{array}\right\} \quad \textit{where} \quad \left\{\begin{array}{l} r\ge 0 \\ \theta\in[0,\pi] \\ \varphi\in[0,2\pi) \end{array}\right. .$$

The Jacobian determinant $\partial(x,y,z)/\partial(r,\theta,\varphi)$ *of the transform is*

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\
&= \cos\theta \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} \\
&= \cos\theta\big(r^2\cos\theta\sin\theta\big) + r\sin\theta\big(r\sin^2\theta\big) \\
&= r^2\sin\theta\,; \\
\Rightarrow \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}$$

as $r \geq 0$ *and* $\sin\theta \geq 0$ *if* $\theta \in [0,\pi]$. *Hence, noting that* $x^2 + y^2 = r^2\sin^2\theta$,

$$\begin{aligned}
\int_\Omega \left(x^2+y^2\right)\mathrm{d}V &= \int_0^\ell \left(\int_0^\pi \left(\int_0^{2\pi} \left(r^2\sin^2\theta\right)\mathrm{d}\varphi\right)\,\sin\theta\,\mathrm{d}\theta\right) r^2\mathrm{d}r \\
&= \int_0^\ell \left(\int_0^\pi \left(2\pi r^2\sin^2\theta\right)\,\sin\theta\,\mathrm{d}\theta\right) r^2\mathrm{d}r \\
&= \int_0^\ell \left(\frac{8}{3}\pi r^2\right) r^2\mathrm{d}r \\
&= \frac{8}{15}\pi\ell^5\,.
\end{aligned}$$

For clarity (for fixed r): in our conventions for spherical polar coordinates the angle θ governs the displacement from the origin in the z-direction whereas ϕ determined the distance from the origin in the $x-y$ plane. We present a series of plots of the surfaces swept out for a variety of angular regions in figure 5.

We provide a summary of these findings and another common coordinate system in table 1.

Table 1: Some common coordinate systems

	Coordinate system	Coordinates	Measure	Notes
Planar	Cartesian	x, y	$\mathrm{d}x\,\mathrm{d}y$	$x, y \in \mathbb{R}$
	Polar	$x = r\cos\theta$ $y = r\sin\theta$	$\mathrm{d}x\,\mathrm{d}y = r\mathrm{d}r\,\mathrm{d}\theta$	$r \geq 0$ $\theta \in [0, 2\pi)$
Volume	Cartesian	x, y, z	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$	$x, y, z \in \mathbb{R}$
	Polar	$x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$ $z = r\cos\theta$	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$	$r \geq 0$ $\theta \in [0, \pi]$ $\varphi \in [0, 2\pi)$
	Cylindrical	$x = r\cos\theta$ $y = r\sin\theta$ z	$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$	$r \geq 0$ $\theta \in [0, 2\pi)$ $z \in \mathbb{R}$

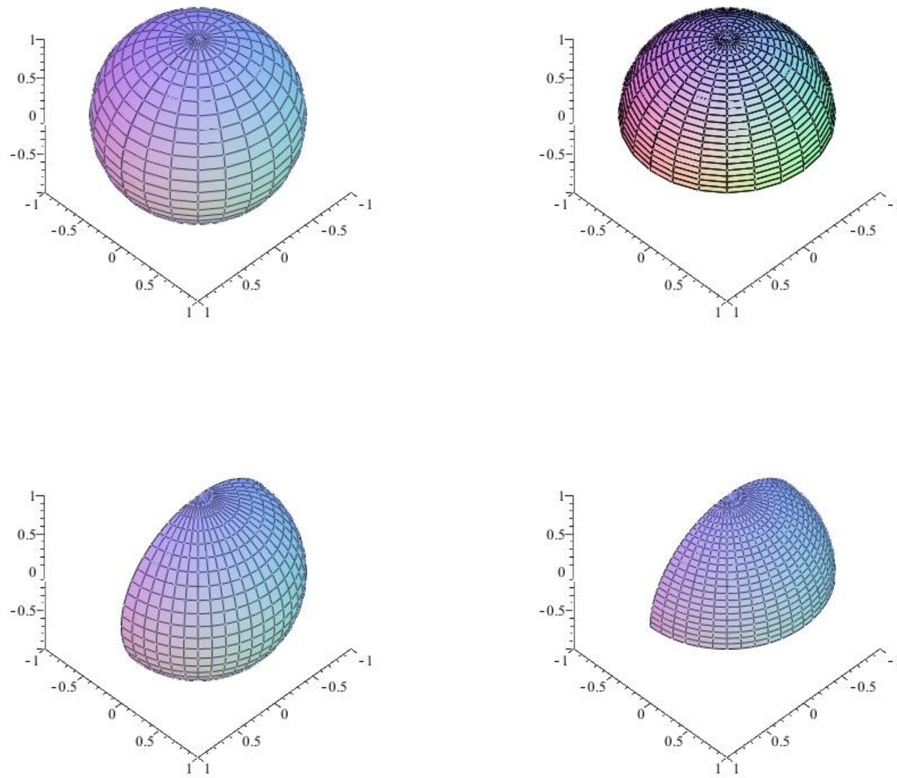

Figure 5: Spherical polar coordinates. $\theta \in [0, \pi), \varphi \in [0, 2\pi]$ (top left), $\theta \in [0, \pi/2), \varphi \in [0, 2\pi)$ (top right), $\theta \in [0, \pi], \varphi \in [0, \pi]$ (bottom left) and $\theta \in [0, \pi/2], \varphi \in [0, \pi]$ (bottom right)

Surface integrals

Table 1 also enables us to infer some results about surface integrals. A surface integral is one of the form $\oint_{\partial\Omega} f(x, y)\,\mathrm{d}S$ where $\partial\Omega$ is a surface bounding the region Ω and $\mathrm{d}S$ is the surface element of $\partial\Omega$. In general $\mathrm{d}S \neq \mathrm{d}x\,\mathrm{d}y$ (i.e., a small patch of area on $\partial\Omega$ is not always a rectangle). One can become convinced of this by considering the following well known phenomenon/brand of chocolate orange: it is impossible to smoothly wrap a sheet of paper around a ball (the fact that the surface of the ball is curved means that you need to introduce a fold).

Rather than integrating over a planar region using Cartesian coordinates we could use planar polar coordinates to integrate a function over $\mathbb{R}^2$. Planar polar coordinates provide us with a surface (circle) of radius r (which we can integrate our function over) before subsequently integrating over all r and thereby sweeping out the entire plane as shown in figure 6.

Figure 6 encourages the following interpretation of the area integral:

$$\int_{\mathbb{R}^2} f(x, y)\,\mathrm{d}A = \underbrace{\int_0^\infty \left(\overbrace{\int_0^{2\pi} f(r, \theta)\, r\mathrm{d}\theta}^{\text{first integrate } f \text{ over the surface of a circle of radius } r} \right) \mathrm{d}r}_{\text{then sweep out the plane by integrating over the radial direction } r}\,.$$

Hence, we may interpret $r\mathrm{d}\theta$ as the surface element of the circle and $\int_0^{2\pi} f(r, \theta)\, r\mathrm{d}\theta$ as the integral of

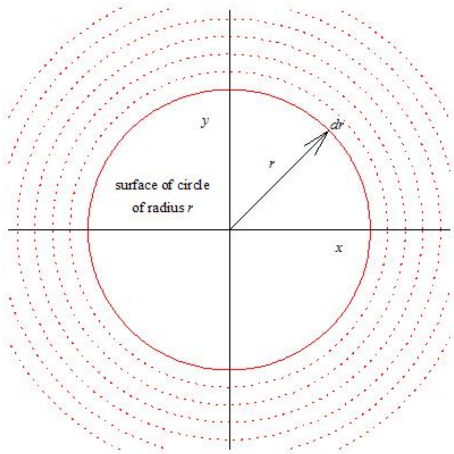

Figure 6: Sweeping out the plane using the surfaces of a series of circles

f over the surface of a circle of (fixed) radius r. Denoting this surface $\partial\mathcal{R}$ we write this integral as $\oint_{\partial\mathcal{R}} f(x,y)\,\mathrm{d}S$.

❏ **Example.** *Let $\ell > 0$ and $\partial\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = \ell^2\}$. Consider the surface integral*

$$\oint_{\partial\mathcal{R}} \left(x + y^2 + 1\right) \mathrm{d}S = \int_0^{2\pi} \left(\ell\cos\theta + \ell^2\sin^2\theta + 1\right)\ell\mathrm{d}\theta = \pi\ell\left(2 + \ell^2\right).$$

Using a similar argument to that of figure 6 we may infer that $r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ is the appropriate surface element for the sphere of radius r.

❏ **Example.** *Let $\ell > 0$ and $\partial\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \ell\}$ be the surface of the sphere of radius ℓ. Consider the integral $\oint_{\partial\Omega} x\,\mathrm{d}S$. In spherical polar coordinates $\mathrm{d}S = \ell^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and the value of x, restricted to the surface $\partial\Omega$, is given by $x = \ell\sin\theta\cos\varphi$. Hence,*

$$\int_{\partial\Omega} x\,\mathrm{d}S = \int_0^{2\pi}\left(\int_0^{\pi}(\ell\sin\theta\cos\varphi)\ell^2\sin\theta\,\mathrm{d}\theta\right)\mathrm{d}\varphi = \int_0^{2\pi}\left(\frac{1}{2}\pi\ell^3\cos\varphi\right)\mathrm{d}\varphi = 0\,.$$

Readers interested in learning how to construct the surface element $\mathrm{d}S$ for other surfaces are directed to the literature (we confess that the technique we have employed here takes advantage of the special geometry considered and does not generalise).

A brief digression on dimensional analysis

The Cartesian coordinates (x,y,z) each have the fundamental dimension of length L. Let f be a physical quantity, then we denote the dimension of f by $\{f\}_{\mathrm{dim}}$. For example, $\{\text{length}\}_{\mathrm{dim}} = \{\text{radius}\}_{\mathrm{dim}} = L$, $\{\text{area}\}_{\mathrm{dim}} = L^2$ and $\{\text{volume}\}_{\mathrm{dim}} = L^3$. Scalars (numbers) are dimensionless which we write $\{\text{scalar}\}_{\mathrm{dim}} = 1$; for example, $\{\text{angle}\}_{\mathrm{dim}} = 1$. Pertinent to our discussion are the observations that

$$\{\mathrm{d}x\}_{\mathrm{dim}} = \{\mathrm{d}r\}_{\mathrm{dim}} = L\,, \quad \{\mathrm{d}x\,\mathrm{d}y\}_{\mathrm{dim}} = L^2\,, \quad \{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\}_{\mathrm{dim}} = L^3 \quad \text{and} \quad \{\mathrm{d}\theta\}_{\mathrm{dim}} = \{\mathrm{d}\varphi\}_{\mathrm{dim}} = 1\,.$$

If we consider the change of coordinates and the Jacobian determinant then

$$\begin{aligned}\{\mathrm{d}x_1\ldots\mathrm{d}x_n\}_{\mathrm{dim}} &= \left\{\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(t_1,\ldots,t_n)}\right|\mathrm{d}t_1\ldots\mathrm{d}t_n\right\}_{\mathrm{dim}} \\ &= \left\{\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(t_1,\ldots,t_n)}\right|\right\}_{\mathrm{dim}}\{\mathrm{d}t_1\}_{\mathrm{dim}}\ldots\{\mathrm{d}t_n\}_{\mathrm{dim}}\end{aligned}$$

Suppose we did not know $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$ we could at least deduce, on purely dimensional grounds, that $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = c(\theta,\varphi)\,r^2\mathrm{d}r\,\mathrm{d}\theta, \mathrm{d}\varphi$ where $\{c(\theta,\varphi)\}_{\mathrm{dim}} = 1$. Indeed, we may sensibly assume that

$$\begin{aligned}
\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = k(r,\theta,\varphi)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \quad &\Leftrightarrow \quad \{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\}_{\mathrm{dim}} = \{k(r,\theta,\varphi)\}_{\mathrm{dim}}\{\mathrm{d}r\}_{\mathrm{dim}}\{\mathrm{d}\theta\}_{\mathrm{dim}}\{\mathrm{d}\varphi\}_{\mathrm{dim}} \\
&\Leftrightarrow \quad L^3 = \{k(r,\theta,\varphi)\}_{\mathrm{dim}} L \\
&\Leftrightarrow \quad L^2 = \{k(r,\theta,\varphi)\}_{\mathrm{dim}} \\
&\Leftrightarrow \quad \{k(r,\theta,\varphi)\}_{\mathrm{dim}} = \{r^2 c(\theta,\varphi)\}_{\mathrm{dim}} \quad \text{where} \quad \{c(\theta,\varphi)\}_{\mathrm{dim}} = 1
\end{aligned}$$

as previously asserted. (Astute readers will have noticed that this argument relies on the implicit assumption that we have not omitted a dimensionful quantity). Hence, had we (erroneously) concluded earlier that $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$ we would know immediately that we had blundered. Finally, we note that this technique cannot provide any insight into the dimensionless quantities but is a useful tool in our armory nonetheless.

The divergence theorem

Gauss' *divergence theorem* is a generalisation of the fundamental theorem of calculus. We present the planar version of this theorem (it's generalisation to higher dimensions requires a discussion of surface integrals and the interested reader is directed to the literature). Let $\mathcal{R} \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\mathcal{R}$ having outward unit normal $\hat{\boldsymbol{n}}$ and $\boldsymbol{F} \in V(\mathcal{R}, \mathbb{R}^2)$ then

$$\int_{\mathcal{R}} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}A = \oint_{\partial\mathcal{R}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s\,.$$

Physically one may interpret $\oint_{\partial\mathcal{R}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s$ as a measure of the amount of $\boldsymbol{F}$ flowing through the boundary $\partial\mathcal{R}$ in the direction $\hat{\boldsymbol{n}}$; consequently, one sometimes refers to such terms as 'flux integrals'. We remark that some authors write $\hat{\boldsymbol{n}}\mathrm{d}s = \mathrm{d}\boldsymbol{s}$ and the divergence theorem as $\int_{\mathcal{R}} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}A = \oint_{\partial\mathcal{R}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{s}$. Rightly or wrongly we shall not adopt this convention for this discussion.

❑ **Example.** *Let $\ell > 0$, $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le \ell^2\}$ and consider $\int_{\mathcal{R}} \mathrm{d}A$ which we have already shown to be equal to $\pi\ell^2$. This may be verified using the divergence theorem in the following way: define $\boldsymbol{F} : \mathcal{R} \mapsto \mathbb{R}^2$ by $(x,y) \mapsto \big(x,y\big)^{\mathrm{T}}/2$ then $\boldsymbol{\nabla}\cdot\boldsymbol{F} = 1$. Moreover, the boundary $\partial\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = \ell^2\}$ and $\hat{\boldsymbol{n}} = \big(x,y\big)^{\mathrm{T}}/\ell$. Therefore, the divergence theorem asserts that*

$$\int_{\mathcal{R}} \mathrm{d}A = \oint_{\partial\mathcal{R}} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}s = \frac{1}{2\ell}\oint_{\partial\mathcal{R}} \big(x^2+y^2\big)\,\mathrm{d}s = \frac{1}{2\ell}\oint_{\partial\mathcal{R}} \ell^2\,\mathrm{d}s = \frac{\ell}{2}\oint_{\partial\mathcal{R}} \mathrm{d}s = \frac{\ell}{2}\int_0^{2\pi} \ell\,\mathrm{d}\theta = \pi\ell^2$$

as expected.

It is straightforward to generalise the divergence theorem to higher dimensions (although we restrict ourselves to the physically meaningful three). Let $\Omega \subset \mathbb{R}^3$ be a bounded region with boundary $\partial\Omega$ having outward unit normal $\hat{\boldsymbol{n}}$ and $\boldsymbol{F} \in V(\Omega, \mathbb{R}^3)$ then

$$\int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V = \oint_{\partial\Omega} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}A$$

where $\mathrm{d}V$ is the volume element $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ (in Cartesian coordinates).

❑ **Example.** *Let $\ell > 0$, $\Omega \subset \mathbb{R}^3$ be the solid sphere of radius ℓ, i.e.*

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le \ell^2\}\,,$$

with normal $\boldsymbol{n} = \boldsymbol{\nabla}\big(x^2+y^2+z^2-\ell\big) = \big(2x, 2y, 2z\big)^{\mathrm{T}}$. *Therefore,* $\hat{\boldsymbol{n}} = \big(x, y, z\big)^{\mathrm{T}}/\ell$. *Define the vector field* $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}$ *by* $(x, y, z) \mapsto \big(x, y, -z\big)^{\mathrm{T}}$ *then the flux integral*

$$\begin{aligned}\oint_{\partial\Omega} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S &= \oint_{\partial\Omega} \begin{pmatrix} x \\ y \\ -z \end{pmatrix} \cdot \begin{pmatrix} x/\ell \\ y/\ell \\ z/\ell \end{pmatrix} \mathrm{d}S \\ &= \frac{1}{\ell}\oint_{\partial\Omega} \big(x^2+y^2-z^2\big)\,\mathrm{d}S \\ &= \frac{1}{\ell}\oint_{\partial\Omega} \big(2x^2+2y^2-\ell^2\big)\,\mathrm{d}S\,.\end{aligned}$$

There are many ways to parametrise the surface $\partial\Omega$ *of the sphere* Ω. *Perhaps the most conventional is*

$$x = \ell\cos\theta\cos\varphi\,,\quad y = \ell\cos\theta\sin\varphi \quad \textit{and} \quad z = \ell\sin\theta \quad \forall\theta\in[0,\pi]\,,\quad \varphi\in[0,2\pi)\,.$$

The surface element $\mathrm{d}S = \ell\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. *Therefore,*

$$\oint_{\partial\Omega} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S = \frac{1}{\ell}\oint_{\partial\Omega}\big(2x^2+2y^2-\ell^2\big)\,\mathrm{d}S = \ell^2\int_0^{2\pi}\left(\int_0^{\pi}\sin^3\theta\,\mathrm{d}\theta\right)\mathrm{d}\varphi = \frac{4\ell^2}{3}\int_0^{2\pi}\mathrm{d}\varphi = \frac{8\pi\ell^2}{3}\,.$$

❑ **Example.** *Let* $\ell > 0$, $\Omega\subset\mathbb{R}^3$ *be the solid cylinder of radius* ℓ *and unit height, i.e.*

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le\ell^2 \textit{ and } z\in[0,1]\}\,,$$

and define $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *by* $(x, y, z) \mapsto \big(xy^2, x^2y, z^2(x^2+y^2)\big)^{\mathrm{T}}$. *Then the flux* $\int_\Omega \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S$ *of* $\boldsymbol{F}$ *flowing out of the cylinder* Ω, *i.e. across* $\partial\Omega$, *is easily calculated using the divergence theorem:*

$$\begin{aligned}\oint_{\partial\Omega} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S &= \int_\Omega \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V \\ &= \int_\Omega \big(x^2+y^2\big)\big(1+2z\big)\,\mathrm{d}V \\ &= \int_0^1\left(\int_0^\ell\left(\int_0^{2\pi} r^2\big(1+2z\big)\,\mathrm{d}\theta\right) r\mathrm{d}r\right)\mathrm{d}z \\ &= \int_0^1\left(\int_0^\ell 2\pi r^2\big(1+2z\big)\,r\mathrm{d}r\right)\mathrm{d}z \\ &= \int_0^1 \frac{\pi\ell^4}{2}\big(1+2z\big)\,\mathrm{d}z \\ &= \pi\ell^4\,.\end{aligned}$$

We close this section with a brief remark about partial differential equations. It is straightforward to establish the identity $\boldsymbol{\nabla}\cdot\big(f\boldsymbol{\nabla}g\big) = f\triangle g + \big(\boldsymbol{\nabla}f\big)\cdot\big(\boldsymbol{\nabla}g\big)$ for any suitably defined scalar fields f, g; an immediate corollary of this observation is that $\boldsymbol{\nabla}\cdot\big(f\boldsymbol{\nabla}g - g\boldsymbol{\nabla}f\big) = f\triangle g - g\triangle f$. Hence, it follows from the divergence theorem that

$$\int_\Omega \big(f\triangle g - g\triangle f\big)\,\mathrm{d}V = \int_{\partial\Omega}\big(f\boldsymbol{\nabla}g - g\boldsymbol{\nabla}f\big)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S\,.$$

This result is known as *Green's identity*. Green's identity is of central importance to the study of partial differential equations where one routinely wishes to construct objects called Green's functions (which one can think of as the inverse of PDE IVP/BVP's).

Green's theorem

Let $\mathcal{R} \subset \mathbb{R}^2$ be a bounded region with boundary $\partial\mathcal{R}$ then: *Green's theorem* in the plane for the field $\boldsymbol{F} : \mathcal{R} \to \mathbb{R}^2$ defined by $(x, y) \mapsto \big(F_1(x, y), F_2(x, y)\big)^{\mathrm{T}}$ is

$$\int_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \oint_{\partial\mathcal{R}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$

where $F_1, F_2 : \mathbb{R}^2 \to \mathbb{R}$ are scalar fields.

❏ **Example.** *Let* $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ *and* $\boldsymbol{F} : \mathcal{R} \to \mathbb{R}^2$ *be defined by* $(x, y) \mapsto \big(x+y, x\big)^{\mathrm{T}}$ *then*

$$\int_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A = \int_{\mathcal{R}} \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x+y)\right) \mathrm{d}A = \int_{\mathcal{R}} (0)\, \mathrm{d}A = 0\,.$$

Next we remark that $\partial\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ *is parametrised by* $t \mapsto \big(\cos(2\pi t), \sin(2\pi t)\big)^{\mathrm{T}}$ *where* $t \in [0, 1)$. *Hence,*

$$\begin{aligned}
\oint_{\partial\mathcal{R}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_0^1 \begin{pmatrix} \cos(2\pi t) + \sin(2\pi t) \\ \cos(2\pi t) \end{pmatrix} \cdot \begin{pmatrix} -2\pi \sin(2\pi t) \\ 2\pi \cos(2\pi t) \end{pmatrix} \mathrm{d}t \\
&= 2\pi \int_0^1 \big(-\sin(2\pi t)\cos(2\pi t) - \sin^2(2\pi t) + \cos^2(2\pi t)\big)\, \mathrm{d}t \\
&= 2\pi \int_0^1 \left(-\frac{1}{2}\sin(4\pi t) + \cos(4\pi t)\right) \\
&= 0
\end{aligned}$$

thereby verifying the statement of Green's theorem in the plane.

❏ **Example.** *Let* $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1$ *and* $y \geq 0\}$ *and* $\boldsymbol{F} : \mathcal{R} \to \mathbb{R}^2$ *be defined by* $(x, y) \mapsto \big(x, -xy\big)^{\mathrm{T}}$ *then*

$$\begin{aligned}
\int_{\mathcal{R}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathrm{d}A &= \int_{\mathcal{R}} \left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial y}(x)\right) \mathrm{d}A \\
&= \int_{\mathcal{R}} (-y)\, \mathrm{d}A \\
&= -\int_0^1 \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, \mathrm{d}x\right) \mathrm{d}y \\
&= -\int_0^1 2y\sqrt{1-y^2}\, \mathrm{d}y \\
&= -\frac{2}{3}
\end{aligned}$$

Next we remark that $\partial\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1$ *and* $y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 \mid x \in [-1, 1]\}$. *Hence,*

$$\begin{aligned}
\oint_{\partial\mathcal{R}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_0^{\frac{1}{2}} \begin{pmatrix} \cos(2\pi t) \\ -\cos(2\pi t)\sin(2\pi t) \end{pmatrix} \cdot \begin{pmatrix} -2\pi \sin(2\pi t) \\ 2\pi \cos(2\pi t) \end{pmatrix} \mathrm{d}t + \int_{-1}^1 \begin{pmatrix} s \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}s \\
&= -2\pi \int_0^{\frac{1}{2}} \left(\frac{1}{2}\sin(4\pi t) + \cos^2(2\pi t)\sin(2\pi t)\right) \mathrm{d}t + \int_{-1}^1 s\, \mathrm{d}s \\
&= -\frac{2}{3}
\end{aligned}$$

thereby verifying the statement of Green's theorem in the plane.

Stokes' theorem

Stokes' theorem generalises Green's theorem in the following way: let $\Omega \subset \mathbb{R}^3$ be a surface (i.e. dimension 2) with normal vector $\hat{\boldsymbol{n}}$ and boundary $\partial\Omega$. Let $\boldsymbol{F} : \Omega \to \mathbb{R}^2$ then

$$\int_\Omega \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \oint_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \,.$$

In the interests of clarify: this result is more correctly known as Kelvin's formulation of Stokes' theorem or the Kelvin-Stokes theorem; the full Stokes theorem requires the machinery of forms from differential geometry and is beyond the scope of this discussion.

❑ **Example.** *Let $\ell > 0$ and $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = \ell^2 \text{ and } z \geq 0\}$ be the surface of the upper hemi-sphere of radius ℓ and outward normal $\hat{\boldsymbol{n}} = \big(x,y,z\big)^{\mathrm{T}}/\ell$. The boundary, $\partial\Omega$, of Ω is the circle given by $\partial\Omega = \{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2 = \ell^2\}$. The vector field $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $(x,y,z) \mapsto \big(-z, y, x\big)^{\mathrm{T}}$ has curl $\boldsymbol{\nabla} \times \boldsymbol{F} = \big(0,-2,0\big)^{\mathrm{T}}$. Therefore, $\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{n}} = -2y$ and*

$$\int_\Omega \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \int_0^{2\pi} \left(\int_0^{\pi/2} \big(-2\ell \sin\theta \sin\varphi\big) \ell^2 \sin\theta \, \mathrm{d}\theta \right) \mathrm{d}\varphi = \int_0^{2\pi} \left(-\frac{1}{2}\pi\ell^3 \sin\varphi \right) \mathrm{d}\varphi = 0 \,.$$

Additionally, we compute $\oint_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$ using the parametrisation:

$$x = \ell\cos\theta\,, \quad y = \ell\sin\theta \quad \text{and} \quad z = 0 \quad \Leftrightarrow \quad \boldsymbol{r} = \ell\big(\cos\theta, \sin\theta, 0\big)^{\mathrm{T}} \quad \Rightarrow \quad \boldsymbol{r'} = \ell\big(-\sin\theta, \cos\theta, 0\big)^{\mathrm{T}}$$

Therefore, $\boldsymbol{r} = \ell\big(\cos\theta, \sin\theta, 0\big)^{\mathrm{T}}$ from which we readily compute

$$\int_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_0^{2\pi} \boldsymbol{F} \cdot \boldsymbol{r'} \, \mathrm{d}\theta = \ell \int_0^{2\pi} \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \mathrm{d}\theta = \ell \int_0^{2\pi} \sin\theta\cos\theta \, \mathrm{d}\theta = 0$$

thereby verifying the statement of Stokes' theorem.

❑ **Example.** *Let $\ell > 0$ and $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = \ell^2 \text{ and } z \geq 0\}$ be the surface of the upper hemi-sphere of radius ℓ and outward normal $\hat{\boldsymbol{n}}$. The boundary, $\partial\Omega$, of Ω is given by $\partial\Omega = \{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2 = \ell^2\}$ (note that this boundary is also the boundary of the disc $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq \ell^2\}$ whose normal vector is $\hat{\boldsymbol{k}}$). Define $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by $(x,y,z) \mapsto \big(y^2\cos(xz), x^3e^{yz}, -e^{-xyz}\big)^{\mathrm{T}}$ then, by two applications of the Kelvin-Stokes theorem,*

$$\int_\Omega \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \oint_{\partial\Omega} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{\mathcal{D}} \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{k}} \, \mathrm{d}A \,.$$

We evaluate this right most integrand:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F} &= \big(xze^{-xyz} - x^3ye^{yz}, -xy^2\sin(xz) - yze^{-xyz}, 3x^2e^{yz} - 2y\cos(xz)\big)^{\mathrm{T}} \\ \Rightarrow \qquad \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{k}} &= 3x^2e^{yz} - 2y\cos(xz) \\ \Rightarrow \qquad \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{k}}\Big|_{\mathcal{D}} &= 3x^2 - 2y \quad (\textit{recall that } z = 0 \textit{ on } \mathcal{D})\,. \end{aligned}$$

Hence,

$$\int_\Omega \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \int_{\mathcal{D}} (3x^2 - 2y) \mathrm{d}A = \int_0^\ell \left(\int_0^{2\pi} (3r^2\cos^2\theta - 2r\sin\theta) \, \mathrm{d}\theta \right) r\mathrm{d}r = \int_0^\ell 3\pi r^3 \, \mathrm{d}r = \frac{3}{4}\pi\ell^4 \,.$$

Acknowledgements

We are grateful to Prof Mark Matsen, Prof Peter Grindrod, Dr Alex Lukyanov, Dr Peter Chamberlain & Dr Nick Biggs for their contributions to this document. Any errors which remain are, of course, the fault of the author.