# Vector Calculus —MA2VC/MA3VC 2016–17— Summary and comparison of the different integrals of fields

The seven types of integrals we have considered in the lectures are in the red boxes.
Those in blue are the formulas we use to compute integrals over curvilinear domains of integration (paths $\Gamma$ and surfaces $S$) as integrals over flat domains (intervals $(t_I, t_F)$ and regions $R$) of the same dimension, by using the parametrisations $\vec{\mathbf{a}} : (t_I, t_F)$ and $\vec{\mathbf{X}} : R \to S$.
The most important theorems relating these kinds of integrals are mentioned in green.
$\hat{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{a}}}{dt} / \left|\frac{d\vec{\mathbf{a}}}{dt}\right|$ and $\hat{\boldsymbol{n}} = \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w} / \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right|$ are the *orientations* of paths and (parametric) surfaces, namely unit tangent and normal vector fields, respectively.

| | | | |
|---|---|---|---|
| **Integrals of real functions** $\int_{(t_I,t_F)} G(t)\,\mathrm{d}t$ <br> Fundamental theorem of calculus applies here. | **Line integrals of scalar fields** $\int_\Gamma f\,\mathrm{d}s$ $= \int_{(t_I,t_F)} f(\vec{\mathbf{a}}) \left|\frac{d\vec{\mathbf{a}}}{dt}\right| \mathrm{d}t$ | **Line integrals of vector fields** $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ $= \int_\Gamma (\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}})\,\mathrm{d}s = \int_{(t_I,t_F)} \vec{\mathbf{F}}(\vec{\mathbf{a}}) \cdot \frac{d\vec{\mathbf{a}}}{dt}\,\mathrm{d}t$ <br> Fundamental theorem of vector calculus applies here. | 1D sets |
| **Double integrals** $\iint_R f\,\mathrm{d}A$ <br> (Polar coordinates can be used.) <br> Green's th. | **Surface integrals of scalar fields** $\iint_S f\,\mathrm{d}S$ $= \iint_R f(\vec{\mathbf{X}}) \left|\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\right| \mathrm{d}A$ | **Fluxes = surface integrals of vector fields** $\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$ $= \iint_S (\vec{\mathbf{F}} \cdot \hat{\boldsymbol{n}})\,\mathrm{d}S = \iint_R \vec{\mathbf{F}}(\vec{\mathbf{X}}) \cdot \frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w}\,\mathrm{d}A$ <br> Stokes' th. | 2D sets |
| **Triple integrals** $\iiint_D f\,\mathrm{d}V$ <br> (Cylindrical or spherical coordinates can be used.) | Divergence th. | | 3D sets |

*Flat* domains of integration: defined by boundary only.

*Curvilinear* domains of integration: parametrisation (either $\vec{\mathbf{a}}$ or $\vec{\mathbf{X}}$) is needed.
Parametrisation allows to compute integrals over flat domains of integration: $(t_I, t_F)$ in place of $\Gamma$, $R$ in place of $S$.

*Scalar* integrands $f, G$: no need of orientation.

*Vector* integrands $\vec{\mathbf{F}}$: orientation ($\hat{\boldsymbol{\tau}}$ or $\hat{\boldsymbol{n}}$) of path/surface needed.

- $G : \mathbb{R} \to \mathbb{R}$ is a real function
- $f : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field
- $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field

These are the *integrands*, namely the functions/fields *of* which we compute integrals.

- $(t_I, t_F) \subset \mathbb{R}$ is an **interval** (1D)
- $R \subset \mathbb{R}^2$ is a planar **region** (2D)
- $D \subset \mathbb{R}^3$ is a **domain** (3D)
- $\Gamma \subset \mathbb{R}^3$ is a **path** (1D), parametrised by $\vec{\mathbf{a}} : (t_I, t_F) \to \Gamma$
- $S \subset \mathbb{R}^3$ is a **surface** (2D), parametrised by $\vec{\mathbf{X}} : R \to S$

These are the *domains of integration*, namely the geometric objects *on* which we compute integrals.

# Vector Calculus —MA2VC/MA3VC 2016–17— Summary and comparison of main vector calculus theorems

| | *The integral on a(n)* | *of* | *of a* | *is equal to the* | *of the* | *on/at the* | *Equivalently, in formulae,* |
|---|---|---|---|---|---|---|---|
| Fundamental theorem of calculus | interval $(t_I, t_F)$ | the derivative | real function $G$ | difference of the values | function $G$ | endpoints | $\int_{t_I}^{t_F} G'(t)\,\mathrm{d}t = G(t_F) - f(t_I)$ |
| Fundamental theorem of vector calculus | oriented path $\Gamma$ from $\vec{\mathbf{p}}$ to $\vec{\mathbf{q}}$ | $\hat{\boldsymbol{\tau}}\cdot$ gradient | scalar field $f$ | difference of the values | field $f$ | endpoints $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ | $\int_\Gamma \vec{\nabla} f \cdot \mathrm{d}\vec{\mathbf{r}} = f(\vec{\mathbf{q}}) - f(\vec{\mathbf{p}})$ |
| Green's theorem | two-dimensional region $R$ | $\hat{\boldsymbol{k}} \cdot$ curl | vector field $\vec{\mathbf{F}}$ | circulation | field $\vec{\mathbf{F}}$ | boundary $\partial R$ | $\iint_R \hat{\boldsymbol{k}} \cdot (\vec{\nabla} \times \vec{\mathbf{F}})\,\mathrm{d}A = \oint_{\partial R} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ |
| Stokes' theorem | oriented surface $(S, \hat{\boldsymbol{n}})$ | $\hat{\boldsymbol{n}}\cdot$ curl | vector field $\vec{\mathbf{F}}$ | circulation | field $\vec{\mathbf{F}}$ | boundary $\partial S$ | $\iint_S (\vec{\nabla} \times \vec{\mathbf{F}}) \cdot \mathrm{d}\vec{\mathbf{S}} = \int_{\partial S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ |
| Divergence theorem | 3D domain $D$ | divergence | vector field $\vec{\mathbf{F}}$ | flux | field $\vec{\mathbf{F}}$ | boundary $\partial D$ | $\iiint_D (\vec{\nabla} \cdot \vec{\mathbf{F}})\,\mathrm{d}V = \oiint_{\partial D} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}}$ |
| | 1st domain of integration | differential operator | function or field | integral type or evaluation | function or field | 2nd domain of integration | formula |