OXFORD — COMPUTATIONAL MATHEMATICS AND APPLICATIONS SEMINAR — 15TH MAY 2014

Plane Wave DG Methods: Exponential Convergence of the hp-version

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The Helmholtz equation

Simplest model of linear & time-harmonic waves:

$$-\Delta u - \omega^2 u = 0$$

in bdd. $\Omega \subset \mathbb{R}^N,\ N = 2, 3,\ \omega > 0,$
(+ impedance/Robin b.c.)

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Why is it interesting?

1. Very general, related to any linear wave phenomena:

wave equation: $\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$
time-harmonic regime: $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})e^{-i\omega t}\}$
$\Bigg\} \to$ Helmholtz equation;

2. plenty of applications;
3. easy to write. . .

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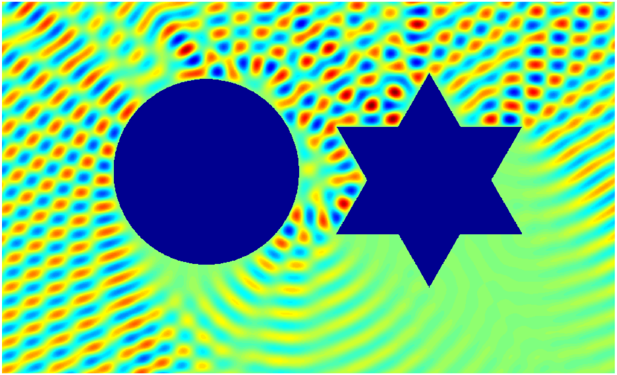
$\Bigg\} \rightarrow$ Helmholtz equation;

2. plenty of applications;
3. easy to write. . . but difficult to solve numerically ($\omega \gg 1$):

- oscillating solutions $\rightarrow$ approximation issue,
- numerical dispersion / pollution effect $\rightarrow$ stability issue.

Difficulty #1: oscillations

Time-harmonic solutions are inherently oscillatory: a lot of DOFs needed for any polynomial discretisation!



(Helmholtz BVP, picture by T. Betcke)

Wavenumber $\omega = 2\pi/\lambda$ is the crucial parameter (λ=wavelength).

Difficulty #2: pollution effect

Big issue in FEM solution for high wavenumbers: pollution effect

$$\frac{\left|\left|\text{Galerkin error}\right|\right|}{\left|\left|\text{best approximation error}\right|\right|} \geq C\,\omega^{a} \qquad a > 0, \quad \omega \to \infty.$$

It affects every (low order) method in h: (BABUŠKA, SAUTER 2000).

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$$\Downarrow$$

Oscillating solutions + pollution effect
= standard FEM are too expensive at high frequencies!

Special schemes required, p- and hp-versions preferred.

ZIENKIEWICZ, 2000: *"Clearly, we can consider that this problem remains unsolved and a completely new method of approximation is needed to deal with the very short-wave solution."*

Trefftz methods

Piecewise polynomials used in FEM are "general purpose" functions, can we use discrete spaces tailored for Helmholtz?

Yes: Trefftz methods are finite element schemes such that test and trial functions are solutions of the Helmholtz equation in each element K of the mesh $\mathcal{T}_h$, e.g.:

$$V_p \subset T(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : \ -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\}.$$

Main idea: more accuracy for less DOFs.

Typical Trefftz basis functions for Helmholtz

1 plane waves (PWs), $\mathbf{x} \mapsto e^{i\omega \mathbf{x}\cdot\mathbf{d}}$ $\mathbf{d} \in \mathbb{S}^{N-1}$

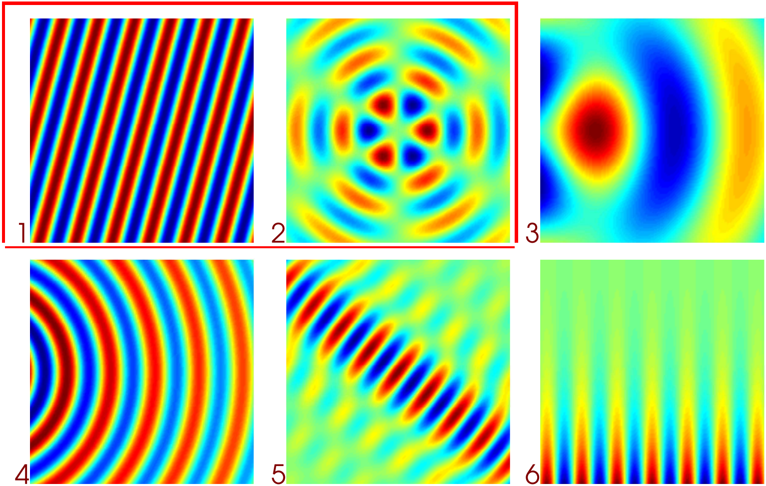
2 circular / spherical waves (CWs),

3 corner waves,

4 fundamental solutions/multipoles,

5 wavebands,

6 evanescent waves, . . .



Wave-based methods

Trefftz schemes require discontinuous functions.
How to "match" traces across interelement boundaries?

Plenty of Trefftz schemes for Helmholtz, Maxwell and elasticity:

- Least squares: method of fundamental solutions (MFS), wave-based method (WBM);
- Lagrange multipliers: discontinuous enrichment (DEM);
- Partition of unity method (PUM/PUFEM), non-Trefftz;
- Variational theory of complex rays (VTCR);
- Discontinuous Galerkin (DG): Ultraweak variational formulation (UWVF).

We are interested in a family of Trefftz-discontinuous Galerkin (TDG) methods that includes the UWVF of Cessenat–Després.

Outline

- TDG method for Helmholtz:
  formulation and a priori (p-version) convergence
- Approximation theory for plane and spherical waves
- Exponential convergence of the hp-TDG

Part I

TDG method for the Helmholtz equation

TDG: derivation — I

1. Consider Helmholtz equation with impedance (Robin) b.c.:

$$
\begin{aligned}
-\Delta u - \omega^2 u &= 0 && \text{in } \Omega \subset \mathbb{R}^N \text{ bdd., Lip., } N = 2, 3 \\
\nabla u \cdot \mathbf{n} + i\omega u &= g && \in L^2(\partial\Omega);
\end{aligned}
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2. introduce a mesh $\mathcal{T}_h$ on Ω;
3. multiply the Helmholtz equation with a test function v and integrate by parts on a single element $K \in \mathcal{T}_h$:

$$\int_K (\nabla u \cdot \nabla \overline{v} - \omega^2 u \overline{v})\, \mathrm{d}V - \int_{\partial K} (\mathbf{n} \cdot \nabla u) \overline{v}\, \mathrm{d}S = 0;$$

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4. integrate by parts again: ultraweak step

$$\int_K (-u\Delta\overline{v} - \omega^2 u \overline{v}) \, \mathrm{d}V + \int_{\partial K} (-\mathbf{n} \cdot \nabla u \, \overline{v} + u \, \mathbf{n} \cdot \nabla \overline{v}) \, \mathrm{d}S = 0;$$

TDG: derivation — II

5 choose a discrete Trefftz space $V_p(K)$ and replace traces on ∂K with numerical fluxes $\widehat{u}_p$ and $\widehat{\boldsymbol{\sigma}}_p$:

$$u \to u_p \quad \text{(discrete solution)} \qquad \text{in } K,$$
$$u \to \widehat{u}_p, \qquad \frac{\nabla u}{i\omega} \to \widehat{\boldsymbol{\sigma}}_p \qquad \text{on } \partial K;$$

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6. use the Trefftz property: $\forall\, v_p \in V_p(K)$

$$\int_K u_p \underbrace{\overline{(-\Delta v_p - \omega^2 v_p)}}_{=0} \mathrm{d}V + \underbrace{\int_{\partial K} \widehat{u}_p \, \overline{\nabla v_p \cdot \mathbf{n}} \, \mathrm{dS} - \int_{\partial K} i\omega \widehat{\boldsymbol{\sigma}}_p \cdot \mathbf{n} \, \overline{v}_p \, \mathrm{dS} = 0}_{\text{TDG eq. on 1 element}}.$$

Two things to set:
discrete space V_p and numerical fluxes $\widehat{u}_p$, $\widehat{\boldsymbol{\sigma}}_p$.

TDG: the space V_p

The abstract error analysis works for every discrete Trefftz space!

Possible choice: plane wave space $(\{\mathbf{d}_\ell\}_{\ell=1}^p \subset \mathbb{S}^{N-1})$

$$V_p(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : \, v|_K(\mathbf{x}) = \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{x}\cdot\mathbf{d}_\ell}, \; \alpha_\ell \in \mathbb{C}, \; \forall K \in \mathcal{T}_h \right\}.$$

$p :=$ number of basis plane waves (DOFs) in each element.

Numerical fluxes

Choose the numerical fluxes as:

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \frac{1}{i\omega}\{\!\{\nabla_h u_p\}\!\} - \alpha\, [\![u_p]\!]_N \\ \widehat{u}_p = \{\!\{u_p\}\!\} - \beta\, \frac{1}{i\omega}[\![\nabla_h u_p]\!]_N \end{cases} \quad \text{on interior faces,}$$

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \frac{\nabla_h u_p}{i\omega} - (1-\delta)\frac{1}{i\omega}\left(\nabla_h u_p + i\omega u_p\,\mathbf{n} - g\,\mathbf{n}\right) \\ \widehat{u}_p = u_p - \delta\frac{1}{i\omega}\left(\nabla_h u_p \cdot \mathbf{n} + i\omega u_p - g\right) \end{cases} \quad \text{on } \partial\Omega.$$

$\{\!\{\cdot\}\!\}$ = averages, $[\![\cdot]\!]_N$ = normal jumps on the interfaces.

$\alpha,\, \beta > 0,\, \delta \in (0, \frac{1}{2}]$ parameters at our disposal (in $L^\infty(\mathcal{F}_h)$):

- h- or p-version, quasi-uniform meshes:
  α, β, δ independent of ω, h, p; UWVF: $\alpha = \beta = \delta = \frac{1}{2}$.
- hp-version, locally refined mesh: α, β, δ depend on local h, p.

Variational formulation of the TDG

With this fluxes, summing over the elements $K \in \mathcal{T}_h$, the TDG method reads: find $u_p \in V_p(\mathcal{T}_h)$ s.t.

$$\mathcal{A}_h(u_p, v_p) = i\omega^{-1} \int_{\partial\Omega} \delta\, g\, \overline{\nabla_h v_p \cdot \mathbf{n}}\, \mathrm{dS} + \int_{\partial\Omega} (1-\delta) g\, \overline{v_p}\, \mathrm{dS},$$

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$\forall\; v_p \in V_p(\mathcal{T}_h)$, where ($\mathcal{F}_h^I$ = interior skeleton)

$$\begin{aligned}
\mathcal{A}_h(u,v) := & \int_{\mathcal{F}_h^I} \{\!\{u\}\!\} [\![\overline{\nabla_h v}]\!]_N \,\mathrm{d}S && + i\,\omega^{-1} \int_{\mathcal{F}_h^I} \beta\, [\![\nabla_h u]\!]_N [\![\overline{\nabla_h v}]\!]_N \,\mathrm{d}S \\
& - \int_{\mathcal{F}_h^I} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!]_N \,\mathrm{d}S && + i\,\omega \int_{\mathcal{F}_h^I} \alpha\, [\![u]\!]_N \cdot [\![\overline{v}]\!]_N \,\mathrm{d}S \\
& + \int_{\partial\Omega} (1-\delta)\, u\, \overline{\nabla_h v \cdot \mathbf{n}} \,\mathrm{d}S && + i\,\omega^{-1} \int_{\partial\Omega} \delta\, \nabla_h u \cdot \mathbf{n}\, \overline{\nabla_h v \cdot \mathbf{n}} \,\mathrm{d}S \\
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\end{aligned}$$

$u_p \mapsto (\mathrm{Im}\, \mathcal{A}_h(u_p, u_p))^{\frac{1}{2}}$ is a norm on the Trefftz space $\Rightarrow$ $\exists !\, u_p$.

"Unconditional quasi-optimality"

On the Trefftz space

$$T(\mathcal{T}_h) := \left\{ v \in L^2(\Omega) : v_{|K} \in H^2(K),\ -\Delta v - \omega^2 v = 0 \text{ in each } K \in \mathcal{T}_h \right\},$$

$$\left.\begin{array}{l} \forall\, v, w \in T(\mathcal{T}_h) : \\ \operatorname{Im} \mathcal{A}_h(v,v) = |||v|||^2_{\mathcal{F}_h} \\ |\mathcal{A}_h(w,v)| \le 2\, |||w|||_{\mathcal{F}_h^+}\ |||v|||_{\mathcal{F}_h} \end{array}\right\} \Rightarrow \begin{array}{c} \text{quasi-optimality:} \\ |||u - u_p|||_{\mathcal{F}_h} \le 3|||u - v_p|||_{\mathcal{F}_h^+} \\ \forall v_p \in V_p(\mathcal{T}_h) \subset T(\mathcal{T}_h). \end{array}$$

Using norms

$$|||v|||^2_{\mathcal{F}_h} := \omega^{-1} \left\| \beta^{1/2} [\![\nabla_h v]\!]_N \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \alpha^{1/2} [\![v]\!]_N \right\|^2_{0,\mathcal{F}_h^I} + \omega^{-1} \left\| \delta^{1/2} \nabla_h v \cdot \mathbf{n} \right\|^2_{0,\partial\Omega} + \omega \left\| (1-\delta)^{1/2} v \right\|^2_{0,\partial\Omega},$$

$$|||v|||^2_{\mathcal{F}_h^+} := |||v|||^2_{\mathcal{F}_h} + \omega \left\| \beta^{-1/2} \{\!\{ v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + \omega^{-1} \left\| \alpha^{-1/2} \{\!\{ \nabla_h v \}\!\} \right\|^2_{0,\mathcal{F}_h^I} + \omega \left\| \delta^{-1/2} v \right\|^2_{0,\partial\Omega}.$$

TDG p-convergence

Monk–Wang duality technique
$\|w\|_{L^2(\Omega)} \leq C(\omega, h, \Omega, \mathcal{T}_h, \alpha, \beta, \delta)|||w|||_{\mathcal{F}_h} \; \forall w \in T(\mathcal{T}_h)$
$\rightarrow$ quasi-optimality in $L^2(\Omega)$-norm.

Assume for now: best approximation estimates for plane or circular waves (shown later in this talk).

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We obtain (h- and) p-estimates for plane/circular waves (2D):

$$|||u - u_p|||_{\mathcal{F}_h} \leq C(\omega h)\, \omega^{-\frac{1}{2}}\, h^{k-\frac{1}{2}} \left(\frac{\log(p)}{p}\right)^{k-\frac{1}{2}} \|u\|_{k+1,\omega,\Omega}\,,$$

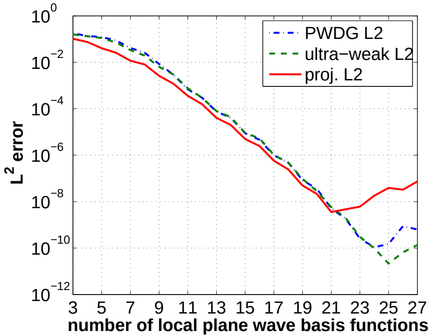
$$\omega\, \|u - u_p\|_{L^2(\Omega)} \leq C(\omega h)\; \mathrm{diam}(\Omega)\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-\frac{1}{2}} \|u\|_{k+1,\omega,\Omega}\,,$$

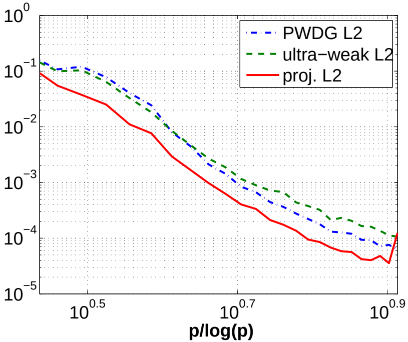
on quasi-uniform meshes with meshsize h.

Slightly different orders of convergence in p in 3D.

Numerical tests

Plane wave spaces, $\omega = 10$, $h = 1/\sqrt{2}$, L^2-norm of errors:





Smooth solution in $C^\infty(\mathbb{R}^2)$

$$u(\mathbf{x}) = J_1(\omega|\mathbf{x}|)\cos\theta$$

exponential convergence.

Singular solution in $H^{\frac{5}{2}-\epsilon}(\Omega)$

$$u(\mathbf{x}) = J_{\frac{3}{2}}(\omega|\mathbf{x}|)\cos(\tfrac{3}{2}\theta)$$

algebraic convergence.

Numerical instability / ill-conditioning for high p!

The road map

	Helmholtz	Maxwell
Formulation of TDG	$\checkmark$	$\sim$ Helm.
TDG $\lvert\lvert\lvert \cdot \rvert\rvert\rvert_{\mathcal{F}_h}$-quasi optimality	$\checkmark$	$\sim$ Helm.
Duality argument	$L^2(\Omega)$	$H(\mathbf{div}, \Omega)'$
hp exponential convergence		
Approximation by GHPs		
Approximation by PWs		

Part II

Approximation in Trefftz spaces

The best approximation estimates

The analysis of any plane wave Trefftz method requires best approximation estimates:

$$-\Delta u - \omega^2 u = 0 \qquad \text{in } D \in \mathcal{T}_h, \qquad u \in H^{k+1}(D),$$

$$\operatorname{diam}(D) = h, \qquad p \in \mathbb{N}, \qquad \mathbf{d}_1, \dots, \mathbf{d}_p \in \mathbb{S}^{N-1},$$

$$\inf_{\vec{\alpha} \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{d}_\ell \cdot \mathbf{x}} \right\|_{H^j(D)} \le C\, \epsilon(h,p)\ \|u\|_{H^{k+1}(D)},$$

with *explicit* $\epsilon(h,p) \xrightarrow[p\to\infty]{h\to 0} 0$.

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with *explicit* $\epsilon(h,p) \xrightarrow[p\to\infty]{h\to 0} 0$.

Goal: precise estimates on $\epsilon(h,p)$

- for plane and circular/spherical waves;
- both in h and p (simultaneously);
- in 2 and 3 dimensions;
- with explicit bounds in the wavenumber ω.

The Vekua theory in N dimensions

We need an old (1940s) tool from PDE analysis: Vekua theory.

$D \subset \mathbb{R}^N$ star-shaped wrt. $\mathbf{0}$, $\omega > 0$.
Define two continuous functions:

$$M_1, M_2 : D \times [0, 1] \to \mathbb{R}$$

$$M_1(\mathbf{x}, t) = -\frac{\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1\big(\omega|\mathbf{x}|\sqrt{1-t}\big),$$

$$M_2(\mathbf{x}, t) = -\frac{i\omega|\mathbf{x}|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1\Big(i\omega|\mathbf{x}|\sqrt{t(1-t)}\Big).$$

The Vekua operators

$$V_1, V_2 : C^0(D) \to C^0(D),$$

$$V_j[\phi](\mathbf{x}) := \phi(\mathbf{x}) + \int_0^1 M_j(\mathbf{x}, t)\phi(t\mathbf{x})\,\mathrm{d}t, \qquad \forall\, \mathbf{x} \in D,\ j = 1, 2.$$

4 properties of Vekua operators

1 $$V_2 = (V_1)^{-1}$$

2 $$\Delta\phi = 0 \qquad \Longleftrightarrow \qquad (-\Delta - \omega^2)\ V_1[\phi] = 0$$

Main idea of Vekua theory:

$$\text{Harmonic functions} \underset{V_1}{\overset{V_2}{\rightleftarrows}} \text{Helmholtz solutions}$$

3 Continuity in (ω-weighted) Sobolev norms, explicit in ω

$(H^j(D),\ W^{j,\infty}(D),\ j \in \mathbb{N})$

4 $$P = \text{Harmonic polynomial} \qquad \Longleftrightarrow \qquad V_1[P] = \text{circular/spherical wave}$$

$$\Big[\underbrace{e^{il\psi}\ J_l(\omega r)}_{2D},\quad \underbrace{Y_l^m\big(\tfrac{\mathbf{x}}{|\mathbf{x}|}\big)\ j_l(\omega|\mathbf{x}|)}_{3D}\Big]$$

Vekua operators & approximation by GHPs

$$-\Delta u - \omega^2 u = 0, \qquad u \in H^{k+1}(D),$$

$$\downarrow V_2$$

$V_2[u]$ is harmonic $\implies$ can be approximated by harmonic polynomials

(harmonic Bramble–Hilbert in h,
$\mathbb{C}$omplex analysis in p-2D (Melenk), new result in p-3D),

$$\downarrow V_1$$

u can be approximated by GHPs:

generalized harmonic polynomials $:= V_1 \left[\begin{array}{c} \text{harmonic} \\ \text{polynomials} \end{array} \right] =$ circular/spherical waves.

($\rightarrow$ Bounds applicable to *any* GHP-based Trefftz schemes!)

The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves:
Jacobi–Anger expansion

2D
$$e^{iz\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(z)\, e^{il\theta} \qquad z\in\mathbb{C},\ \theta\in\mathbb{R},$$

3D
$$\underbrace{e^{ir\boldsymbol{\xi}\cdot\boldsymbol{\eta}}}_{\text{plane wave}} = 4\pi \sum_{l\geq 0}\sum_{m=-l}^{l} i^l \underbrace{j_l(r)\, Y_{l,m}(\boldsymbol{\xi})}_{\text{GHP}}\, \overline{Y_{l,m}(\boldsymbol{\eta})} \qquad \boldsymbol{\xi},\boldsymbol{\eta}\in\mathbb{S}^2,\ r\geq 0.$$

We need the other way round:

GHP $\approx$ linear combination of plane waves

- truncation of J–A expansion,
- careful choice of directions (in 3D),
- solution of a linear system,
- residual estimates,

$\rightarrow$ explicit error bound.

The final approximation by plane waves

$-\Delta u - \omega^2 u = 0$ $\xrightarrow{V_2}$ $-\Delta V_2[u] = 0$

harmonic approx. $\downarrow$

Circular waves $\xleftarrow{V_1}$ Harmonic polyn.

$\downarrow$ (Jacobi–Anger)$^{-1}$

Plane waves

Final estimate

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{\ell=1}^{p} \alpha_\ell e^{i\omega\, \mathbf{x}\cdot \mathbf{d}_\ell} \right\|_{j,\omega,D} \leq C(\omega h)\, h^{k+1-j} q^{-\lambda(k+1-j)} \|u\|_{k+1,\omega,D}$$

In 2D: $p = 2q+1$, $\lambda(D)$ explicit, $\forall\ \mathbf{d}_\ell$.

In 3D: $\underbrace{p = (q+1)^2}_{\text{better than poly.!}}$, $\lambda(D)$ unknown, special $\mathbf{d}_\ell$.

If u extends outside D: exponential order in q. (Same for GHPs.)

The road map

	Helmholtz	Maxwell
Formulation of TDG	✓	$\sim$ Helm.
TDG $\lvert\lvert\lvert \cdot \rvert\rvert\rvert_{\mathcal{F}_h}$-quasi optimality	✓	$\sim$ Helm.
Duality argument	$L^2(\Omega)$	$H(\mathbf{div}, \Omega)'$
hp exponential convergence		
Approximation by GHPs	✓	✓ (p non sharp)
Approximation by PWs	✓	✓ (non sharp)

Part III

What about hp-TDG?

What do we want?

hp-convergence is achieved by combination of mesh refinement and increase of #DOFs per element.

Typical hp-result on a priori graded meshes for Laplace 2D:

$$\left\| u - u_{hp} \right\|_{H^1(\Omega)} \leq C e^{-b \sqrt[3]{\#DOFs}} \qquad C, b > 0.$$

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We prove, for TDG + plane/circular wave basis, Helmholtz 2D:

$$\left\| u - u_{hp} \right\|_{L^2(\Omega)} \leq C e^{-b \sqrt[2]{\#DOFs}} \qquad C, b > 0.$$

What do we need?

Consider 2D Helmholtz impedance (+Dirichlet) BVP,
with piecewise analytic domain Ω and boundary conditions g.

So far we have proved:

- unconditional well-posedness and quasi-optimality,
- approximation bounds in h and p simultaneously.

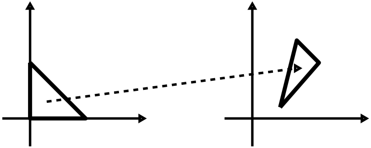
What else do we need to obtain exponential convergence?

- specify meshes and fluxes (modify duality);
- analytic regularity and extendibility of solutions;
- improved approximation bounds. . .

Explicit dependence on element geometry

Polynomial FEM: best approximation bounds on $K \in \mathcal{T}_h$ obtained by scaling to reference element $\hat{K}$.

$\Delta u + \omega^2 u = 0$ in K, $\rightarrow$ pullback $\hat{u}(\hat{x}) := u\big(F(\hat{x})\big)$ is not Trefftz
$\rightarrow$ not approximable by Trefftz basis.



Even for affine scaling:
$\mathbb{P}^q(\hat{K}) \longrightarrow \mathbb{P}^q(K)$
$PW^q(\hat{K}) \longrightarrow ???$

Every element K has "its own" approximation bound
$\rightarrow$ constants depend on the shape of K $\rightarrow$ (in principle) not uniformly bounded on unstructured graded meshes.

We want "universal bounds" independent of the geometry, but. . .

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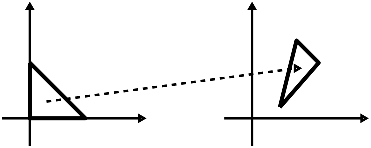

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We want "universal bounds" independent of the geometry, but... we get more: fully explicit bounds for curvilinear non-convex elements.

Assumption and tools

Assumption on element D: (Very weak!)

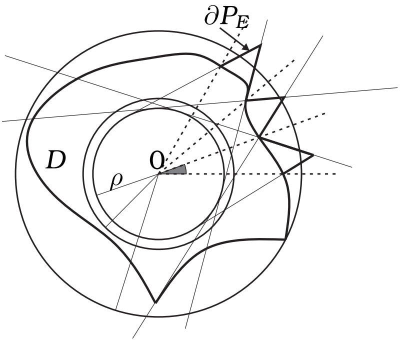
- $D \subset \mathbb{R}^2$ s.t. $\operatorname{diam}(D) = 1$, star-shaped wrt B_ρ, $0 < \rho < 1/2$.

Define:

- $D_\delta := \{z \in \mathbb{R}^2, d(z, D) < \delta\}$, $\quad \xi := \begin{cases} 1 & D \text{ convex}, \\ \frac{2}{\pi} \arcsin \frac{\rho}{1-\rho} < 1. \end{cases}$

Use:

- M. Melenk's ideas;
- complex variable, identification $\mathbb{R}^2 \leftrightarrow \mathbb{C}$, harmonic $\leftrightarrow$ holomorphic;
- conformal map level sets, Schwarz–Christoffel;
- Hermite interpolant q_n;
- lot of "basic" geometry and trigonometry, nested polygons, plenty of pictures...



Explicit approximation estimate

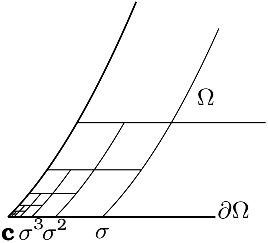
Approximation result

Let $n \in \mathbb{N}$, f holomorphic in D_δ, $0 < \delta \le 1/2$,
$h := \min\left\{(\xi\delta/27)^{1/\xi}/3,\ \rho/4\right\}$, $\Rightarrow$ $\exists q_n$ of degree $\le n$ s.t.

$$\|f - q_n\|_{L^\infty(D)} \le 7\rho^{-2}\, h^{-\frac{72}{\rho^4}} (1+h)^{-n} \|f\|_{L^\infty(D_\delta)}\,.$$

- Fully explicit bound;
- exponential in degree n;
- $h \ge$"conformal distance"$(D, \partial D_\delta)$, related to physical dist. δ;
- in convex case $h = \min\{\delta/27,\ \rho/4\}$;
- extends to harmonic f/q_n and derivatives ($W^{j,\infty}$-norm);
- extended to Helmholtz solutions and circular/plane waves (fully explicit $W^{j,\infty}(D)$-continuity of Vekua operators).

"Geometric meshes"



Sequence of meshes with:

- element diameters h_K geometrically graded (with $0 < \sigma < 1$) towards domain corners;
- any star-shaped element allowed! K star-shaped wrt $B_{\rho h_K}(\mathbf{x}_K)$.

ρ and σ are important parameters in the convergence.

Increase #DOFs by simultaneously:

1. refining layer of small elements,
2. increasing number of PWs/CWs in each element.

The TDG flux parameters

We simply choose the flux parameters $(h_K := \operatorname{diam} K)$

$$\alpha = \mathsf{a}\frac{\max_{K\in\mathcal{T}_h} h_K}{\min\{h_{K_1}, h_{K_2}\}} \quad \text{on } K_1 \cap K_2, \qquad \mathsf{a}, \beta, \delta > 0 \text{ constant.}$$

This choice gives "balance" between approximation and duality.

To guarantee shape-independence, we develop new trace estimates with explicit dependence on the element geometry through the parameter ρ.

Approximation in the elements

Need to bound $\inf_{v_p \in V_p} \|u - v_p\|$ in two cases:

1 Exponentially small elements at domain corners.
Use that in tiny elements PWs / CWs behave like $\mathbb{P}^1$ polynomials.
Difficulty: $\nabla u \notin L^\infty$, $u \notin H^2$.

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2 Larger elements away from corners.
Following Melenk, $u \in \mathcal{B}^2_{\underline{\beta}, \frac{1}{1+\omega}}(\Omega)$, weighted countably-normed space, and extends analytically (similar to Laplace solutions):

$\Rightarrow$ $h_K \sim d(K,\text{corners}) \sim d\big(K, \partial(\text{analyticity region of } u)\big) \quad \forall K.$

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Putting everything together: desired exponential convergence

$$\left\|u - u_{hp}\right\|_{L^2(\Omega)} \leq C e^{-b \sqrt[2]{\#DOFs}} \qquad C, b > 0.$$

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Duality argument	$L^2(\Omega)$	$H(\mathbf{div},\Omega)'$
hp exponential convergence	✓ (2D)	$\times$
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Summary and open problems

What we have done:

- TDG formulation, unconditional well-posedness;
- approximation theory: holomorphic, harmonic, Helmholtz;
- h- and p-convergence for plane and spherical waves;
- exponential hp-convergence on graded meshes in 2D;
- (not discussed: extensions to Maxwell equations).

Plenty of possible research directions:

- non-constant coefficients $\omega(\mathbf{x})$;
- adaptivity in PW directions;
- other PDEs, time-domain;
- new bases;
- defeat ill-conditioning, . . .

Thank you!

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