Integral equation methods for acoustic scattering by fractals

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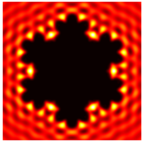

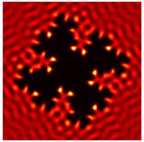

A. Caetano (Aveiro), S.N. Chandler-Wilde (Reading), X. Claeys (LJLL),
A. Gibbs (UCL), D.P. Hewett (UCL)

arXiv:2309.02184 — :IFSintegrals

Acoustic wave scattering

Time-harmonic acoustic waves:
Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^n$, $n \in \{2,3\}$, with wavenumber $k > 0$.

Direct scattering: incoming wave $\underbrace{u^i}_{\text{datum}}$ hits obstacle $\underbrace{\Gamma}_{\text{datum}}$ and generates scattered field $\underbrace{u^s}_{\text{unknown}}$.

Consider Dirichlet (sound-soft) boundary conditions on a bounded Γ.

Acoustic wave scattering

Time-harmonic acoustic waves:
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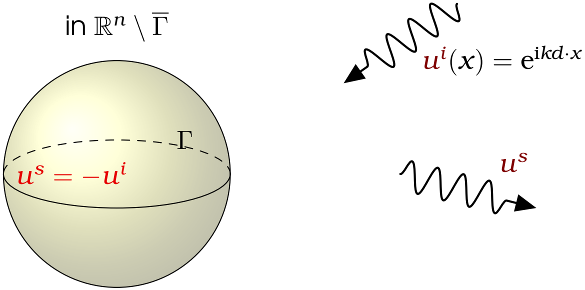
Consider Dirichlet (sound-soft) boundary conditions on a bounded Γ.

$$\Delta u^s + k^2 u^s = 0$$

in $\mathbb{R}^n \setminus \overline{\Gamma}$

$$u^{tot} = u^i + u^s$$

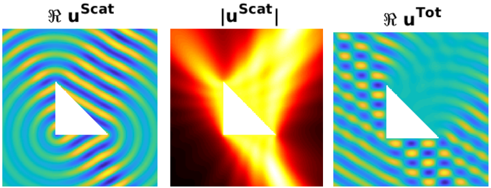


u^s satisfies Sommerfeld radiation condition (SRC) at infinity: $\lim_{r=|x|\to\infty} r^{\frac{n-1}{2}} (\partial_r u^s - iku^s) = 0$

Scattering by Lipschitz domains and screens

Classical problem e.g. when:

1. Γ is the boundary of a Lipschitz domain of $\mathbb{R}^n$



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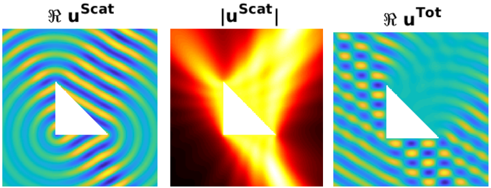

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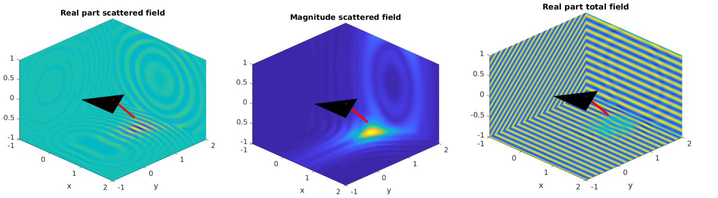
2. Γ is Lipschitz subset of $\{x \in \mathbb{R}^n, x_n = 0\}$ (planar screen)



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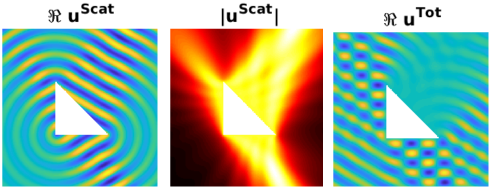

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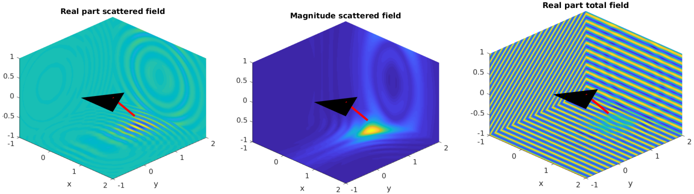

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Neumann trace (jump, in case ②) $\phi = [\partial_n u^s]$ on Γ is solution of single-layer BIE $S\phi = -\gamma u^i$, scattered field represented with layer potential $u^s = \mathcal{S}\phi$. BIE approximated with BEM.

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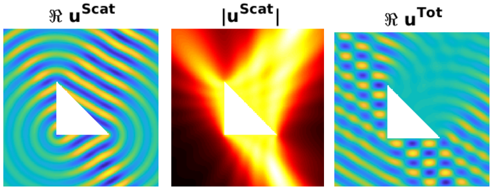

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What happens when Γ is much rougher than this, e.g. fractal?

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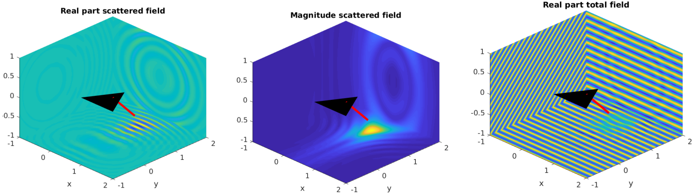

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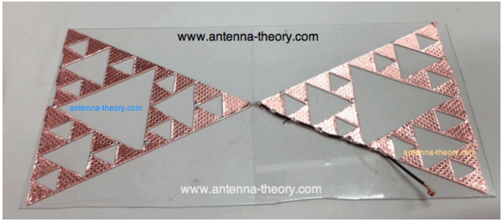
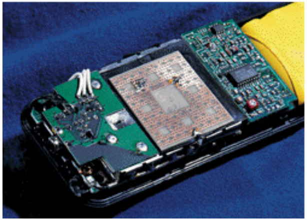
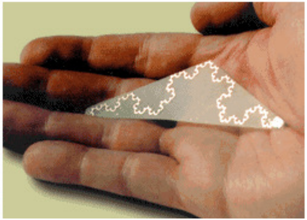
Waves and fractals: applications

Fractals model roughness at multiple scales, in natural and man-made objects:

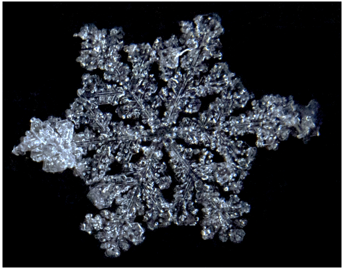






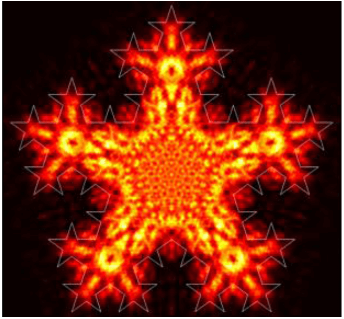
Wideband fractal antennas

▲ http://www.antenna-theory.com/antennas/fractal.php



◀ Scattering by ice crystals in atmospheric physics (C. Westbrook)

Fractal apertures in laser optics (J. Christian) ▶

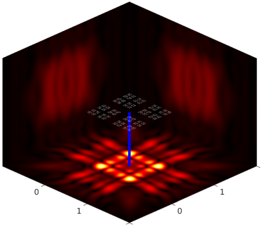


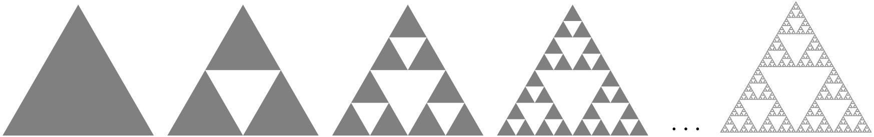
M.V. Berry 1979, "Diffractals": *a new regime in wave physics*

Scattering by fractals

Plenty of mathematical challenges:

- How to formulate well-posed BVPs?
  What is the right function space setting?
  How to write BVP as integral equation?
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?
- How to exploit self-similarity?
- ...





Tools developed here (hopefully!) relevant to (numerical) analysis of
other IEs, ΨDOs, BVPs, numerical integration on rough, complicated, fractal sets.

Our main contributions

This talk: AC, SCW, XC, AG, DH, AM, arXiv:2309.02184
Integral equation methods for acoustic scattering by fractals

BVPs, integral equations, function spaces

- SCW, DH, IEOT, 2015
  Wavenumber-explicit continuity & coercivity est. in acoustic scattering by planar scr.
- SCW, DH, AM, IEOT, 2017
  Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to BIEs on fractal scr.
- SCW, DH, SIAM J. Math. Anal., 2018
  Well-posed PDE and integral equation formulations for scattering by fractal screens,
- AC, DH, AM, JFA, 2021
  Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets

Numerical methods

- SCW, DH, AM, J.Besson, Numer. Math., 2021
  Boundary element methods for acoustic scattering by fractal screens
- J.Bannister, AG, DH, M3AS, 2022
  Acoustic scattering by impedance screens/cracks with fractal boundary. . .
- AG, DH, AM, Numer. Algorithms, 2022
  Numerical quadrature for singular integrals on fractals
- AC, SCW, AG, DH, AM, arXiv:2212.06594, 2022
  A Hausdorff-measure BEM for acoustic scattering by fractal screens
- AG, DH, B.Major Numer. Algorithms, 2023
  Numerical evaluation of singular integrals on non-disjoint self-similar fractal sets

Two ways to apply BEM to fractal Γ — ref.s to flat screen case

1. Chandler-Wilde, Hewett, Moiola, Besson, Numer. Math. 2021

2. Caetano, Chandler-Wilde, Gibbs, Hewett, Moiola, arXiv:2212.06594

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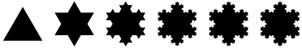
1 Chandler-Wilde, Hewett, Moiola, Besson, Numer. Math. 2021

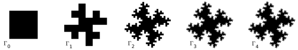
Approximate Γ with Lipschitz "prefractal" Γ_j and apply conventional BEM on each Γ_j

open $\Gamma_j \subset \Gamma_{j+1}$



compact $\Gamma_j \supset \Gamma_{j+1}$

non-nested $\Gamma_j \not\subset \not\supset \Gamma_{j+1}$



- "Non-conforming", since typically $V_N \not\subset V = H_\Gamma^{-1/2}$
- BVP and BEM convergence from Mosco convergence of spaces
- – No convergence rates
- – Requires "thickened prefractals"
- Can use any BEM implementation

2 Caetano, Chandler-Wilde, Gibbs, Hewett, Moiola, arXiv:2212.06594

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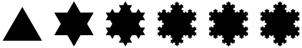

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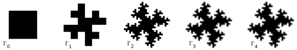

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2 CAETANO, CHANDLER-WILDE, GIBBS, HEWETT, MOIOLA, arXiv:2212.06594

- Discretise Γ without approximation
- Conforming method $V_N \subset V = H_\Gamma^{-1/2}$
- Easy convergence from Céa lemma + rates

} Rest of this talk!

- Integration wrt Hausdorff measure $\mathcal{H}^d$ $\rightarrow$ require special quadrature formulas

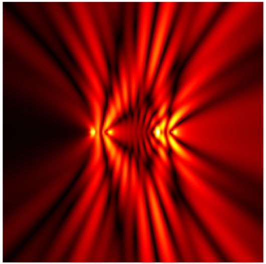
What do we do?

3 levels of generality for Γ

- Arbitrary compact $\Gamma \subset \mathbb{R}^n$:
  BVP, Newton potential & op., variational form
  THEOREM: BVP and IE well-posedness

- d-sets:
  "intrinsic" function spaces, trace operators
  integral operators, piecewise-constant Galerkin
  THEOREM: Galerkin convergence

- IFS attractors:
  tree structure, wavelets, quadrature rule
  THEOREM: Galerkin convergence rates



+ Quadrature rule
+ Numerical results

julia implementation for general class of IFS:
https://github.com/AndrewGibbs/IFSintegrals

Arbitrary compact $\Gamma \subset \mathbb{R}^n$

BVP: $\Delta u^s + k^2 u^s = 0$ in $\Omega := \mathbb{R}^n \setminus \Gamma$, Sommerfeld r.c., $u^s + u^i \in W_0^{1,\mathrm{loc}}(\Omega)$

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Standard acoustic Newton potential $\mathcal{A} : H^s_{\mathrm{comp}}(\mathbb{R}^n) \to H^{s+2}_{\mathrm{loc}}(\mathbb{R}^n)$:

$$\mathcal{A}\psi(x) := \int_{\mathbb{R}^n} \Phi(x,y)\psi(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}^n, \qquad \Phi(x,y) := \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) & n = 2 \\ \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} & n = 3 \end{cases}$$

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Spaces:

$$H_\Gamma^{-1} := \{v \in H^{-1}(\mathbb{R}^n) : \operatorname{supp} v \subset \Gamma\}, \qquad \left(H_\Gamma^{-1}\right)^* = \widetilde{H}^1(\Omega)^\perp$$

$$\widetilde{H}^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1(\mathbb{R}^n)}$$

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$a(\phi,\psi) := \langle A\phi, \psi\rangle_{H^1(\mathbb{R}^n)\times H^{-1}(\mathbb{R}^n)}$ is continuous & compactly-perturb. coercive in $H_\Gamma^{-1} \times H_\Gamma^{-1}$

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THEOREM. Except for possibly countably many k, ($\forall k > 0$ if Ω connected)

- $A := H_\Gamma^{-1} \to \widetilde{H}^1(\Omega)^\perp$ is invertible
- the BVP has unique solution $u^s \in H^{1,\mathrm{loc}}(\mathbb{R}^n)$
- $u^s = \mathcal{A}\phi$ where $\phi \in H_\Gamma^{-1}$ is the unique solution of the IE $A\phi = g$ with $g := -P(\sigma u^i)$

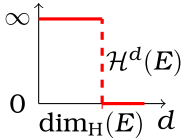
Part I

IE and Galerkin on d-sets

Hausdorff measure and d-sets

Hausdorff measure and dimension of $E \subset \mathbb{R}^n$, $0 \le d \le n$: $(\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E))$

$$\mathcal{H}^d(E) := \lim_{\delta \searrow 0} \inf_{\{U_i\}} \left\{ \sum_{i=1}^{\infty} (\mathrm{diam} U_i)^d : \bigcup_{i=1}^{\infty} U_i \supset E, \mathrm{diam} U_i < \delta \right\}, \quad \dim_{\mathrm{H}}(E) := \inf\{d : \mathcal{H}^d(E) = 0\}$$



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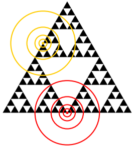

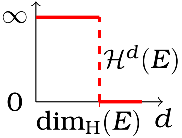

"Uniformly locally d-dimensional sets".

FALCONER, TRIEBEL, JONSSON&WALLIN, ...

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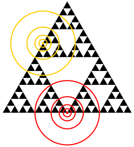
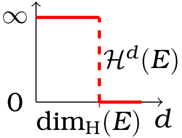
“Uniformly locally d-dimensional sets”.

FALCONER, TRIEBEL, JONSSON&WALLIN, . . .

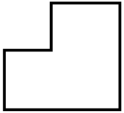




Examples of d-sets in $\mathbb{R}^2$:

(a) Closure of a bounded Lipschitz open set

$d = 2$,



(b) Boundary of a bounded Lipschitz open set

$d = 1$,

(c) Line segment screen

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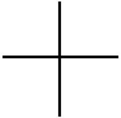
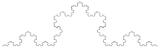


(d) Multi-screen

$d = 1$,

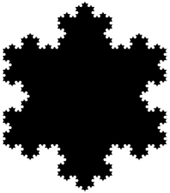
(e) Cantor set screen

$d = \frac{\log 2}{\log 3}$,



(f) Koch curve

$d = \frac{\log 4}{\log 3}$,



(g) Koch snowflake

$d = 2$

d-sets: function spaces and integral operator

On d-set Γ, define $\mathbb{L}_2(\Gamma)$ as the space of square-integrable functions wrt measure $\mathcal{H}^d|_\Gamma$.

Can define "intrinsic" Sobolev spaces $\mathbb{H}^t(\Gamma)$. $\mathbb{H}^t(\Gamma) \subset \mathbb{L}_2(\Gamma) \subset \mathbb{H}^{-t}(\Gamma) = \mathbb{H}^t(\Gamma)^*$, $t > 0$.

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For $s > \frac{n-d}{2}$, it extends to $\mathrm{tr}_\Gamma : H^s(\mathbb{R}^n) \to \mathbb{L}_2(\Gamma)$.

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tr_Γ and its adjoint tr_Γ^* are unitary isomorphisms in: $(n-2 < d \leq n)$

$$
\begin{array}{ccccc}
\mathbb{H}^{1-\frac{n-d}{2}}(\Gamma) & \subset & \mathbb{L}_2(\Gamma) & \subset & \mathbb{H}^{-1+\frac{n-d}{2}}(\Gamma) \\
\mathrm{tr}_\Gamma \uparrow & & & & \downarrow \mathrm{tr}_\Gamma^* \\
\widetilde{H}^1(\mathbb{R}^n \setminus \Gamma)^\perp & & & & H_\Gamma^{-1} \\
\cap & & & & \cap \\
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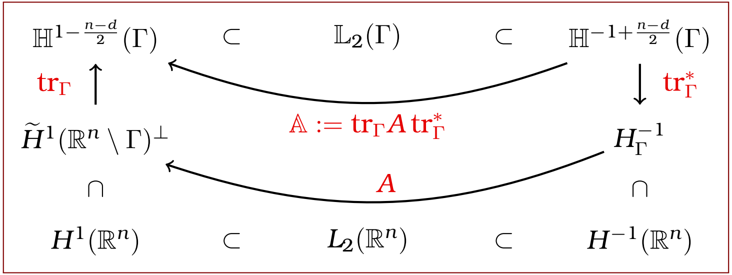

$\mathbb{A}$ is a single-layer operator between $\mathbb{H}^t(\Gamma)$ spaces

THEOREM. $\mathbb{A}$ is an integral operator in Hausdorff measure:

$$\forall \Psi \in L_\infty(\Gamma), \qquad \mathbb{A}\Psi(x) = \int_\Gamma \Phi(x,y)\Psi(y)\, \mathrm{d}\mathcal{H}^d(y) \qquad \mathcal{H}^d\text{-a.e. } x \in \Gamma$$

The Hausdorff-measure integral equation method

Re-write IE $A\phi = g$ and (coercive+compact) variational pr. for $\widetilde{\phi} \in \mathbb{H}^{-t_d}(\Gamma)$, $t_d := 1 - \frac{n-d}{2}$:

$$\mathbb{A}\widetilde{\phi} = \mathrm{tr}_\Gamma g \qquad \iff \qquad \langle \mathbb{A}\widetilde{\phi}, \widetilde{\psi}\rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} = \langle \mathrm{tr}_\Gamma g, \widetilde{\psi}\rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} \quad \forall\widetilde{\psi} \in \mathbb{H}^{-t_d}(\Gamma)$$

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What's the advantage?

We can apply Galerkin method with any N-dimensional $\mathbb{V}_N \subset \mathbb{L}_2(\Gamma) \overset{\text{dense}}{\subset} \mathbb{H}^{-t_d}(\Gamma)$.

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$$\underline{\underline{A}}\vec{c} = \vec{b}, \qquad A_{i,j} = \int_{T_i}\int_{T_j} \Phi(x,y)\, \mathrm{d}\mathcal{H}^d(x)\mathrm{d}\mathcal{H}^d(y), \qquad b_i = -\int_{T_i} g(x)\, \mathrm{d}\mathcal{H}^d(x)$$

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If Γ is boundary of bdd Lipschitz domain, screen or multi-screen (CLAEYS, HIPTMAIR 2013), then this coincides with classical single-layer BIE and BEM, $d = n - 1$.

If Γ is planar d-set, it coincides with (AC, SCW, AG, DH, AM 2022).

Part II

IEM on IFS attractors

Iterated function systems (IFS)

IFS is a family of M contracting similarities: (FALCONER, HUTCHINSON, TRIEBEL,...)

$$s_m : \mathbb{R}^n \to \mathbb{R}^n, \qquad |s_m(x) - s_m(y)| = \rho_m|x - y|, \qquad 0 < \rho_m < 1, \qquad m = 1, \ldots, M.$$

There exists a unique non-empty compact Γ with $\Gamma = s(\Gamma)$, where $s(E) := \bigcup_{m=1}^{M} s_m(E)$.

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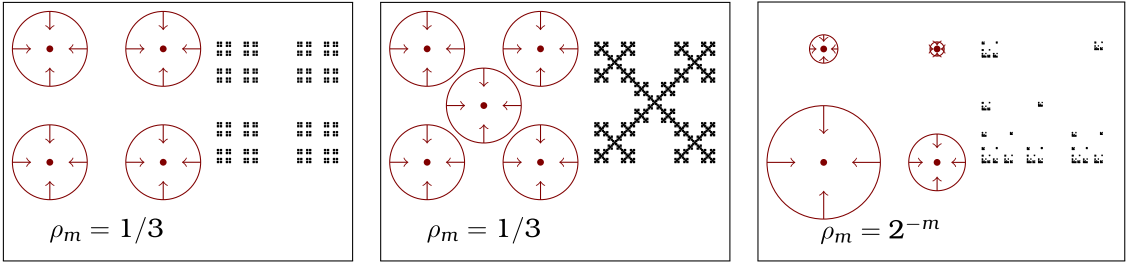

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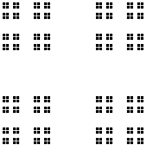
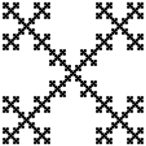
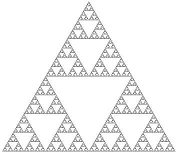


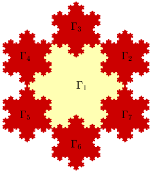

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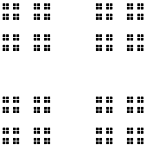
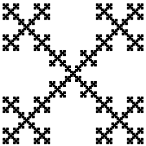
IFS is homogeneous if $\rho_m = \rho \ \forall m$ (then $d = \frac{\log M}{\log 1/\rho}$).

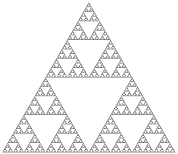
Γ is disjoint if $\Gamma_m := s_m(\Gamma)$ are all disjoint.
Disjoint implies OSC and $d < n$.



$M = 4$, D, H



$M = 5$, ND, H



$M = 3$, ND, H

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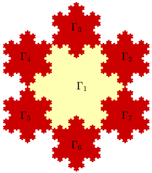
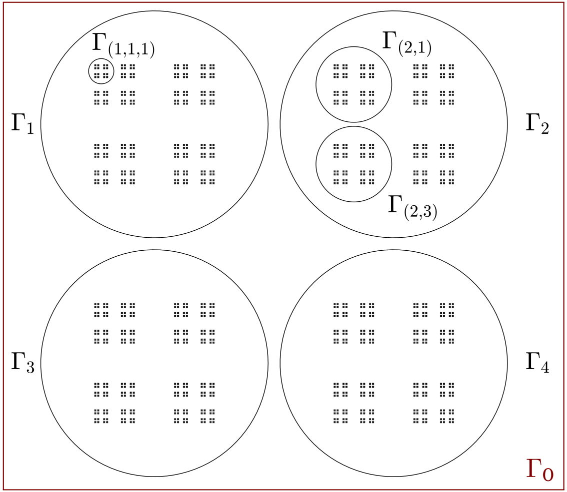


$M = 7$, ND, NH

IFS tree structure and wavelets

Disjoint IFS attractor Γ have natural decompositions in elements $\Gamma_{\mathbf{m}} = s_{m_1} \circ \cdots \circ s_{m_\ell}(\Gamma)$, $\mathbf{m} = (m_1, \ldots, m_\ell) \in \{1, \ldots, M\}^\ell$, $\ell \in \mathbb{N}$, that are similar copies of Γ itself.



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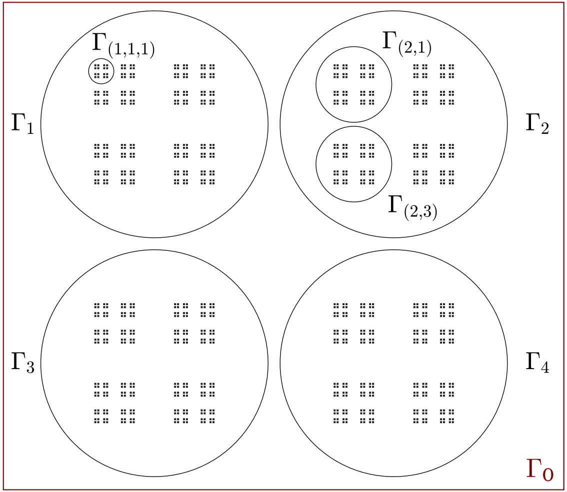

Linear combinations of characteristic functions $\chi_{\mathbf{m}}$ of $\Gamma_{\mathbf{m}}$ give hierarchical orthonormal wavelet basis of $\mathbb{L}_2(\Gamma)$.

Collecting $\Gamma_{\mathbf{m}}$s according to diameter, wavelet basis gives characterisation of $\mathbb{H}^t(\Gamma)$ and its norm. (JONSSON 1998)

We use $\mathrm{span}\{\chi_{\mathbf{m}}\}$ for a suitable partition with $\mathrm{diam}(\Gamma_{\mathbf{m}}) \le h$ as Galerkin space $\mathbb{V}_N$

Piecewise-constant space on IFS attractor

We exploit IFS tree structure to construct Galerkin space and basis: $0 < h < \mathrm{diam}(\Gamma)$

$$\mathbb{V}_N = \mathrm{span}\Big\{\chi_{\mathbf{m}},\ \mathbf{m} \in \{1,\ldots,M\}^{\ell}, \ell \in \mathbb{N},\ \ \mathrm{diam}(\Gamma_{\mathbf{m}}) \le h,\ \ \mathrm{diam}(\Gamma_{(m_1,\ldots,m_{\ell-1})}) > h\Big\} \subset \mathbb{L}_2(\Gamma)$$

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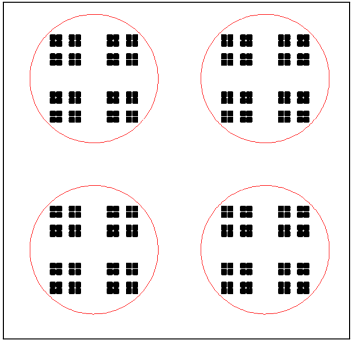
Each element $\Gamma_{\mathbf{m}}$ is a copy of Γ under similarity $s_{\mathbf{m}}$, with $\mathrm{diam}(\Gamma_{\mathbf{m}}) \le h$.

$\mathrm{diam}(\Gamma) = \sqrt{2}$, $M = 4$



$\rho = \frac{1}{3}$, $h = 0.5$, $N = 4$

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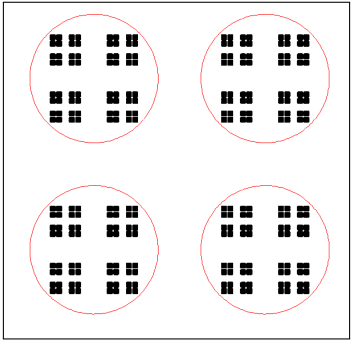

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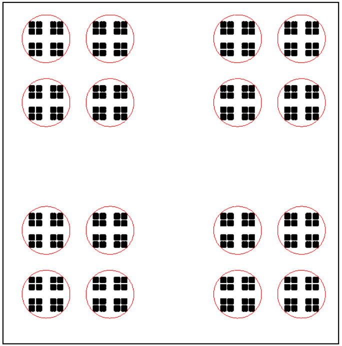

$\rho = \frac{1}{3}$, $h = 0.2$, $N = 16$

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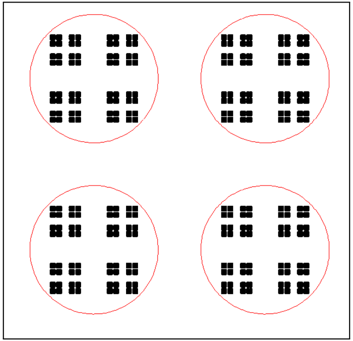

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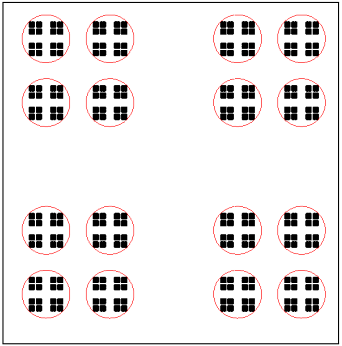

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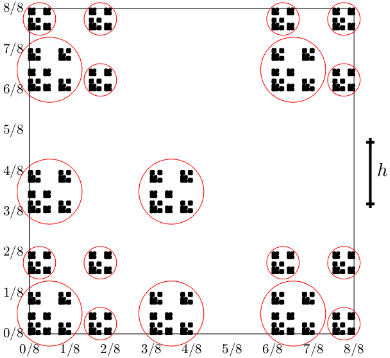

$\rho_1 = \frac{1}{2}$, $\rho_{2:4} = \frac{1}{4}$, $h = 0.2$, $N = 19$

Piecewise-constant IEM convergence for disjoint IFS attractors

Using Fredholm, relation Galerkin space/wavelets, coefficient decay in $\mathbb{H}^t(\Gamma)$:

Theorem (AC, SCW, XC, AG, DH, AM 2023)

Γ disjoint IFS attractor, $n-2 < d = \dim_{\rm H}(\Gamma) < n$.
$\mathbb{V}_N$ piecewise constants on self-similar partition $\{\Gamma_{\mathbf{m}}\}$ of Γ, $\mathrm{diam}(\Gamma_{\mathbf{m}}) \le h$.
Assume IE solution $\phi \in H^s_\Gamma$ for some $-1 < s < -\frac{n-d}{2}$.

Then $$\left\|\widetilde{\phi} - \widetilde{\phi}_N\right\|_{\mathbb{H}^{-1+\frac{n-d}{2}}(\Gamma)} = \|\phi - \phi_N\|_{H^{-1}_\Gamma} \le c\, h^{s+1} \|\phi\|_{H^s_\Gamma}$$

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- h^{2s+2} super-convergence of linear functionals, e.g.: point value $u^s(x)$ and far-field
- No higher regularity (and rate) can be expected: $H_\Gamma^{-\frac{n-d}{2}} = \{0\}$
- For homogeneous IFS ($\rho_m = \rho$), if maximal regularity is achieved, rates are

$$M^{-\ell/2} \quad \text{for } n = 2, \qquad (\rho M)^{-\ell/2} \quad \text{for } n = 3$$

  with ℓ the "level" of the pw-constant space $(h = \rho^\ell \mathrm{diam}(\Gamma),\ N = M^\ell)$
- For $d = n - 1$, we recover classical results for Lipschitz screens and boundaries
  For $\Gamma \subset \{x_n = 0\}$, we recover (AC, SCW, AG, DH, AM 2022)

Part III

Numerical quadrature

Numerical quadrature on IFS attractors

Linear system requires quadrature rule to approximate

$$A_{j,j'} = \int_{\Gamma_{\mathbf{m}(j)}} \int_{\Gamma_{\mathbf{m}(j')}} \Phi(x,y)\, \mathrm{d}\mathcal{H}^d(y)\mathrm{d}\mathcal{H}^d(x), \qquad b_j = -\int_{\Gamma_{\mathbf{m}(j)}} u^i(x)\, \mathrm{d}\mathcal{H}^d(x)$$

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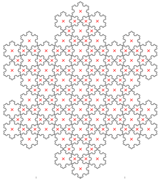
Quadrature rule:

- decompose $\Gamma_{\mathbf{m}}$ in similar sub-components, using IFS structure
- split Helmholtz kernel in Laplace + smoother terms
- exploit Laplace kernel homogeneity and IFS self-similarity to reduce singular integral to a smooth one
- treat smooth integrands with composite barycentre rule, using IFS
- express all singular integrals in terms of a few "fundamental" ones



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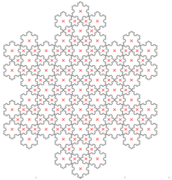


Convergence analysis of quadrature error and of fully discrete Galerkin system.

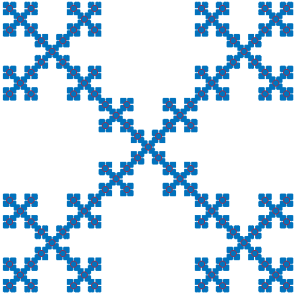
Extend to "invariant measures", more general than Hausdorff (HUTCHINSON 1981).

Each $\Gamma_{\mathbf{m}}$ is similar copy of Γ: for simplicity we just consider integrals over Γ.

Disjoint case: (AG, DH, AM 2022). Non-disjoint case: (AG, DH, B. MAJOR 2023).

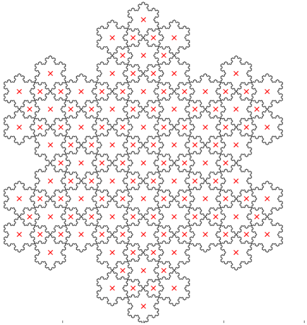
Barycentre rule for smooth integrals

As before, partition Γ in $\Gamma_{\mathbf{m}} = s_{\mathbf{m}}(\Gamma)$ with $\mathrm{diam}(\Gamma_{\mathbf{m}}) \approx h_Q$.



Extend classical midpoint rule:
Approximate $f|_{\Gamma_{\mathbf{m}}}$ with $f(\mathbf{x_m})$, where $x_{\mathbf{m}}$ is barycentre of $\Gamma_{\mathbf{m}}$

$$\int_\Gamma f(x)\mathrm{d}\mu(x) = \sum_{\mathbf{m}} \int_{\Gamma_{\mathbf{m}}} f(x)\mathrm{d}\mu(x) \approx \sum_{\mathbf{m}} \mu(\Gamma_{\mathbf{m}}) f(\mathbf{x_m})$$



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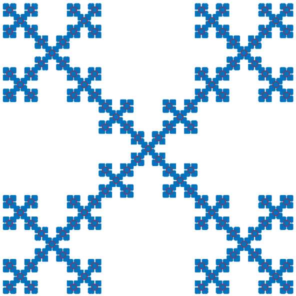

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Barycentre and weights are easily computed:

$$\mu(\Gamma_{\mathbf{m}}) = p_{m_1} \cdots p_{m_\ell} \mu(\Gamma), \qquad p_m = \rho_m^d,$$

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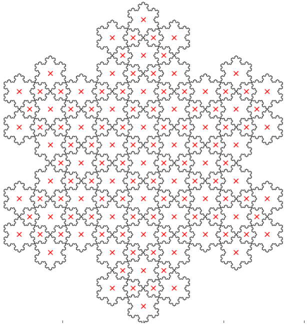

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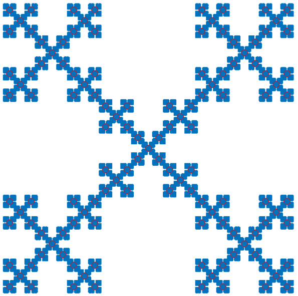
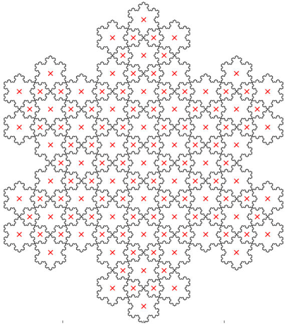
$$\text{Error} \le \frac{n}{2} h_Q^2 \, \mu(\Gamma) \, |f|_{C^2(\bigcup_{\mathbf{m}} \mathrm{Hull}(\Gamma_{\mathbf{m}}))}$$

Same story for double integrals.

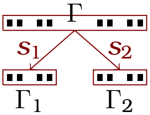




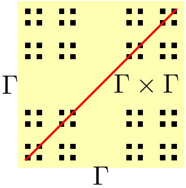
Quadrature rule for singular homogeneous integrals

Integrability. Γ a compact d-set, $y \in \Gamma$:

$$\int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d, \qquad I^t_{\Gamma,\Gamma} := \int_\Gamma \int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(y) \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d.$$



Singularity of $|x-y|^{-t}$ is localised on the red line.



▲ Example:
Cantor set $\subset \mathbb{R}$
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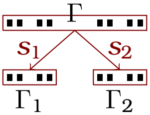

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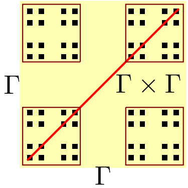
Decompose double integral over $\Gamma \times \Gamma$: $$I^t_{\Gamma,\Gamma} = \sum_{m=1}^{M} \sum_{m'=1}^{M} I^t_{\Gamma_m,\Gamma_{m'}}$$



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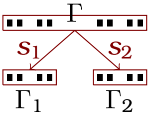

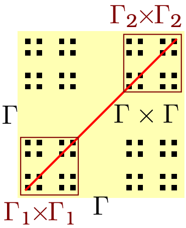

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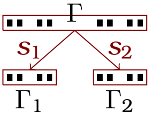

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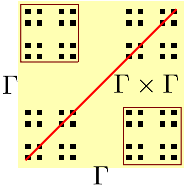

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Can compute $I^t_{\Gamma,\Gamma}$ only in terms of (smooth!) off-diagonal integrals:

$$I^t_{\Gamma,\Gamma} = \frac{1}{1-\sum_{m=1}^M \rho_m^{2d-t}} \sum_{m=1}^{M} \sum_{\substack{m'=1 \\ m' \neq m}}^{M} I^t_{\Gamma_m,\Gamma_{m'}}$$

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Compute $I^t_{\Gamma,\Gamma}$ by applying barycentre rule to smooth $I^t_{\Gamma_m,\Gamma_{m'}}$, $m \neq m'$

All this extends to: $\log|x-y|$, invariant measures $\mu \neq \mu'$, single integrals.

Quadrature and integral equation

Split Helmholtz fundamental solution as

$$\Phi(x,y) = \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) = -\frac{1}{2\pi}\log|x-y| + \mathcal{R}(|x-y|) & \text{in } \mathbb{R}^2 \\ \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} = \frac{1}{4\pi|x-y|} + \mathcal{R}(|x-y|) & \text{in } \mathbb{R}^3 \end{cases} \qquad \mathcal{R} \text{ Lipschitz}$$

Compute the elements of the Galerkin matrix and RHS vector by approximating homogeneous term with self-similar rule and smooth term $\mathcal{R}$ with barycentre rule.

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BEM error bounds taking into account the approximation of the integrals.

h^2 convergence rate is preserved if $h_Q \lesssim h^{1+d}$ ($h_Q \lesssim h^{1+d/2}$ for homogeneous IFS).
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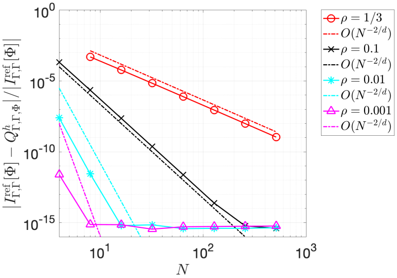
Barycentre rule requires value of $\mathcal{H}^d(\Gamma)$: not known for most fractals $\Gamma \notin \mathbb{R}$!
This is irrelevant for the computation of near-field $u^s(x)$ and far-field in scattering BVP.

Quadrature: numerical examples

Approximation of the integral of the Helmholtz fundamental solution on $\Gamma \times \Gamma$



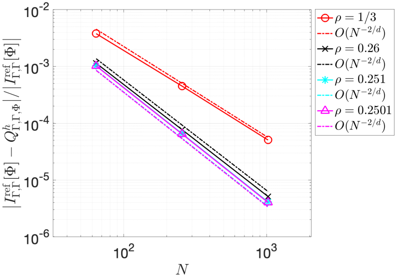
◀ Cantor sets in $\mathbb{R}$

Cantor dusts in $\mathbb{R}^2$ ▶

$k = 5$

Error plotted against
\# quadrature points

Dashed lines
= theoretical rates



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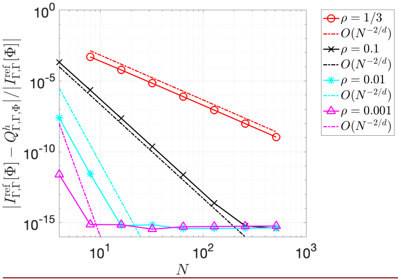

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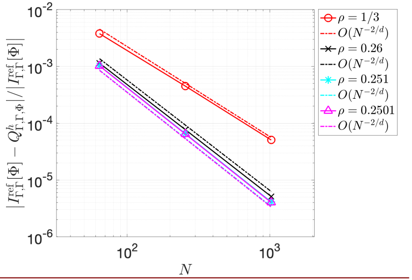
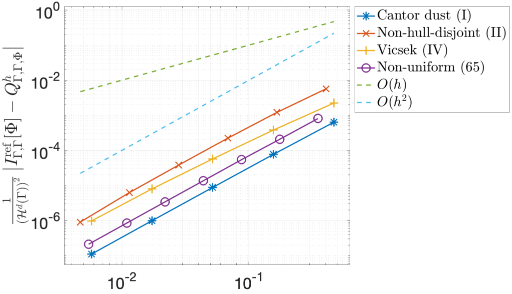
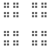
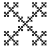





Cantor dust


non “hull-disjoint”


non-disjoint


non-uniform

$k = 2$

Error plotted against $h_{\mathcal{Q}}$

Barycentre rule vs chaos game (Monte Carlo)

Chaos game is alternative quadrature rule: (Forte, Mendivil, Vrscay 1998)

(i) choose $x_0 \in \mathbb{R}^n$

(ii) sequence $\{m_j\}_{j\in\mathbb{N}}$ of i.i.d. random variables in $\{1,\dots,M\}$ with probabilities $\{p_1,\dots,p_M\}$

(iii) construct the stochastic sequence $x_j = s_{m_j}(x_{j-1})$ for $j \in \mathbb{N}$

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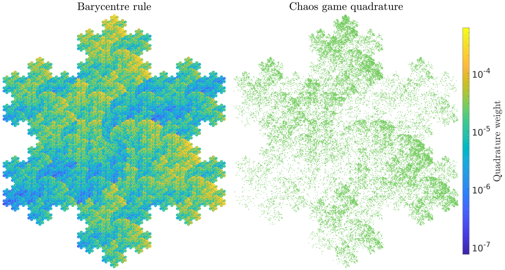
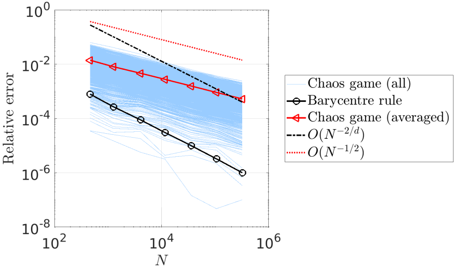
Approximation of $\int_\Gamma f\mathrm{d}\mu$ for $f \in C^\infty$ on $\Gamma =$ Koch snowflake (IFS: $M=7, \rho_{1:6}=\frac{1}{3}, \rho_7=\frac{1}{\sqrt{3}}$)
$\mu =$ invariant measure with non-homogeneous weights p_m. 1000 random realisations.



◀ Nodes & weights (1 realis.)

Relative errors ▶

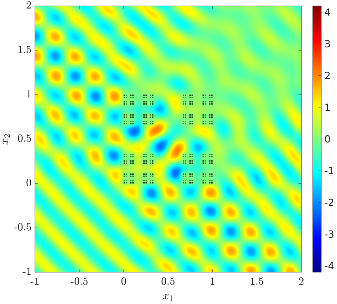


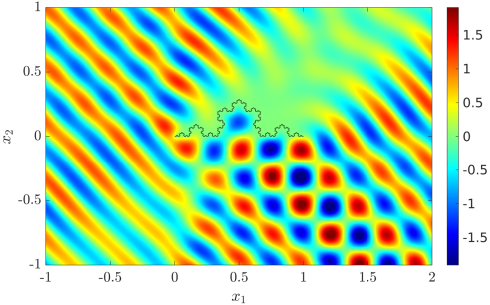
Part IV

Numerics

$n = 2$

Total field for scattering by Cantor dust and Koch curve. $M = 4, \rho = \frac{1}{3}, d = \frac{\log 4}{\log 3}, k = 20.$





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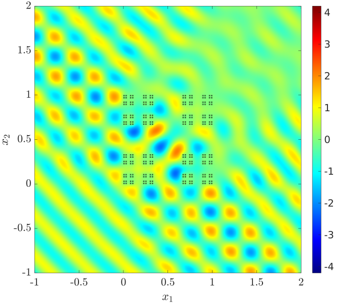
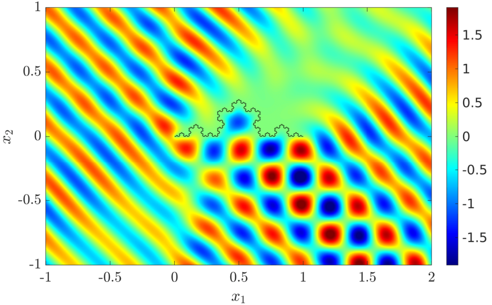
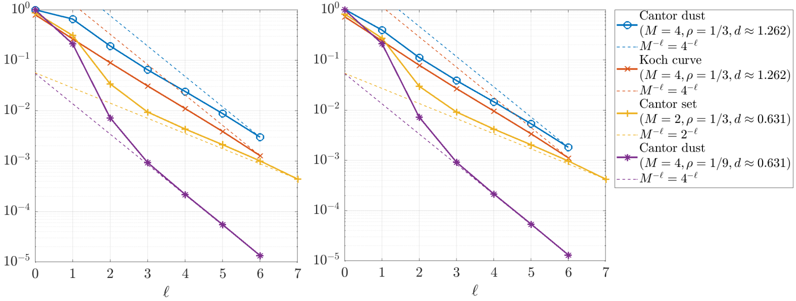


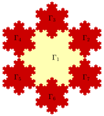


Near- & far-field
relative L_∞ error
for different shapes,
$k = 5$.

Dashed lines $= M^{-\ell}$
conv. rates under
maximal regularity:
achieved for $d \leq 1$

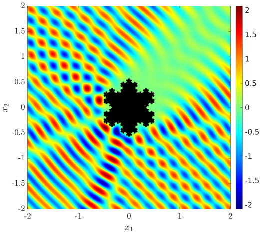


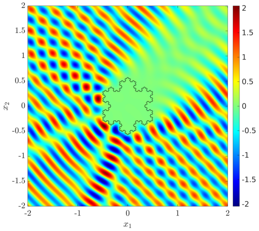
Koch snowflake

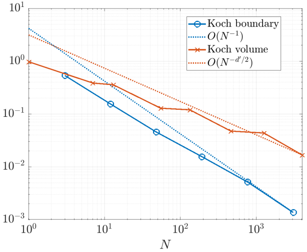


Two ways of approximating the scattering by a Koch snowflake Γ:

1. Γ = closure of open set: non-homog. IFS with $M = 7$, $d = 2$, $\rho_1 = \frac{1}{\sqrt{3}}$, $\rho_{2:7} = \frac{1}{3}$
2. $\partial\Gamma$ = union of 3 Koch curves: 3 IFSs with $M = 4$ each, $d = \frac{\log 4}{\log 3}$, $\rho = \frac{1}{3}$



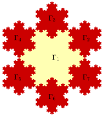




◀ Far-field L_∞ relative error

2 requires that k^2 is not eigenvalue of Γ

Koch snowflake



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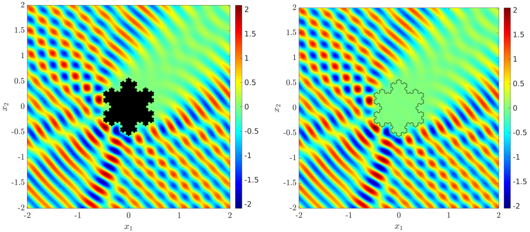
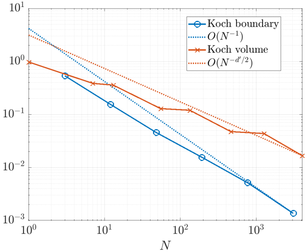


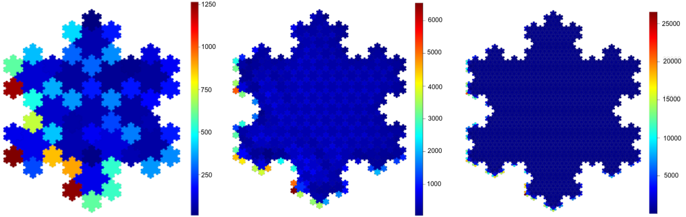


◀ Far-field L_∞ relative error

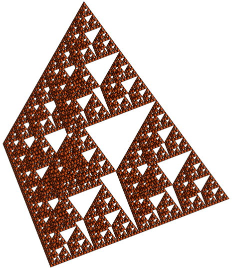
2 requires that k^2 is not eigenvalue of Γ

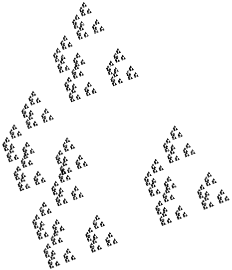
We show that the solution of IE 1 satisfies $\phi \in H^{-1}_{\partial\Gamma} \subset H^{-1}_{\Gamma}$

Refining the mesh, ϕ_N localises on boundary: plot of $|\phi_N|$ ▶



$n = 3$

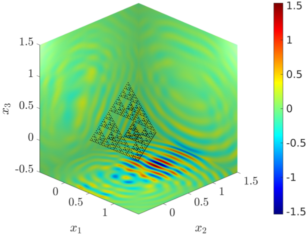


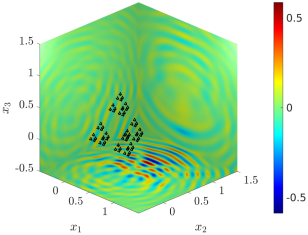


- Sierpinski tetrahedron, $M = 4$.

Left: $\rho = \frac{1}{2}$, $d = 2$, connected

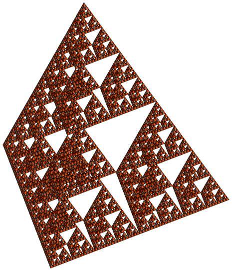
Right: $\rho = \frac{3}{8}$, $d = \frac{\log 4}{\log(8/3)}$, disjoint

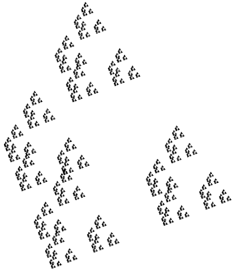




- scattered field, $k = 50$, $\ell = 7$, $N = 16\,384$

$n = 3$





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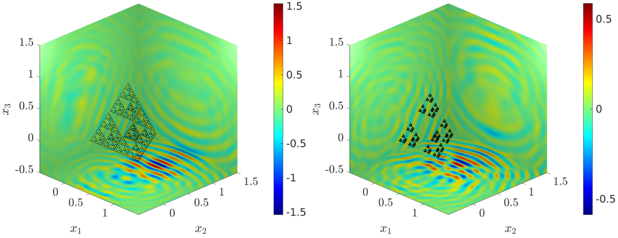
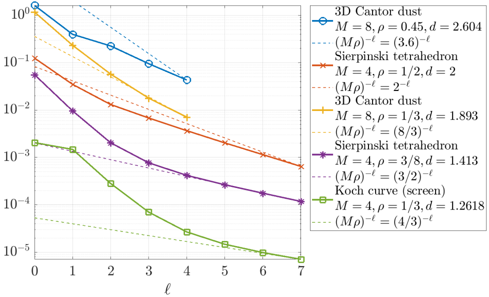
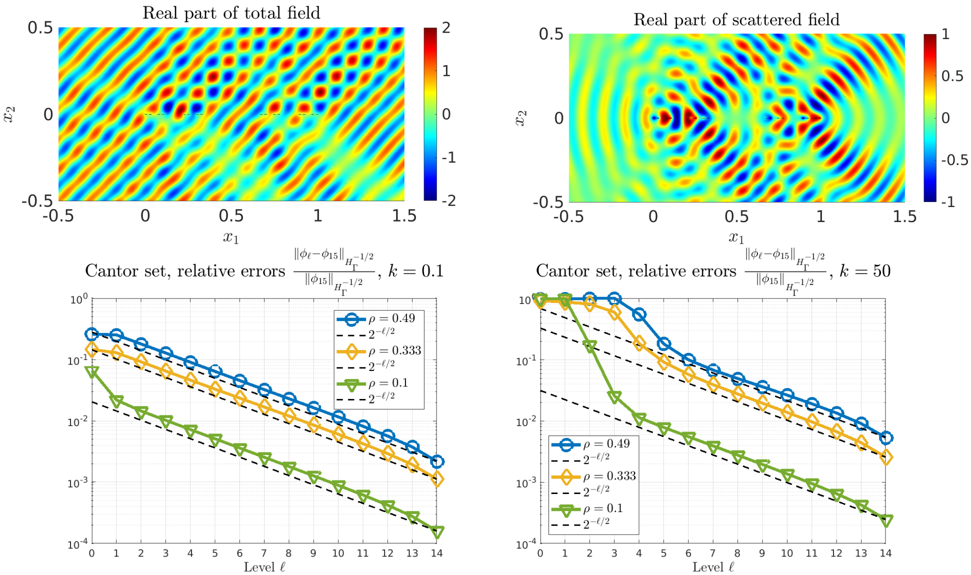




▲ scattered field, $k = 50$, $\ell = 7$, $N = 16\,384$

far-field L_∞ error (increments), $k = 2$ ▲

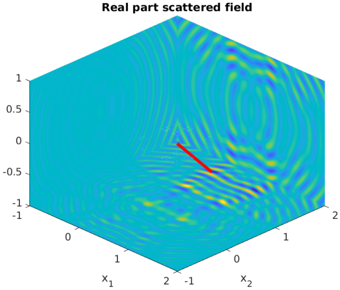
$n = 2$, flat screen: Cantor set $\Gamma \subset \mathbb{R}$

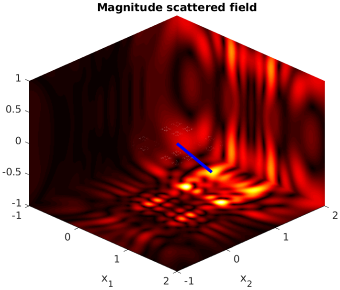


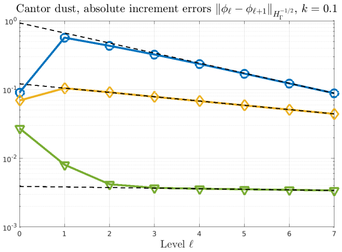
Rate $2^{-\ell/2}$ in $H_\Gamma^{-1/2}$ norm as expected, independent of ρ. $u^i(x) = e^{ik\theta \cdot x}$

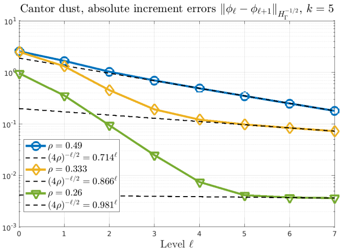
Similar plots (with double rate $2^{-\ell}$) for near-field $u^s(x)$ and far-field.

$n = 3$, flat screen: Cantor dust $\Gamma \subset \mathbb{R}^2$



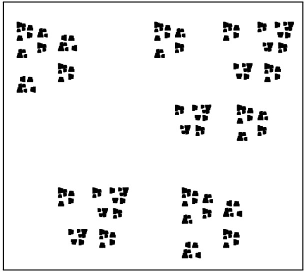


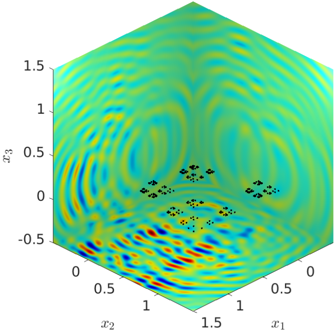


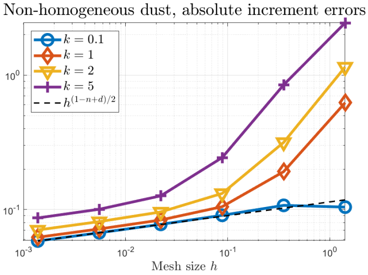


ρ-dependent rate $(4\rho)^{-\ell/2}$ in $H_\Gamma^{-1/2}$ norm as expected.
Double rates $(4\rho)^{-\ell}$ for near-field and far-field.

$n = 3$, flat screen: non-homogeneous dust & Sierpinski triangle

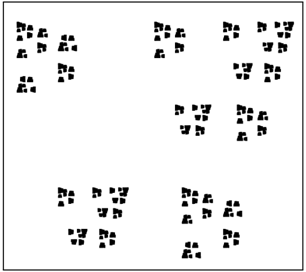


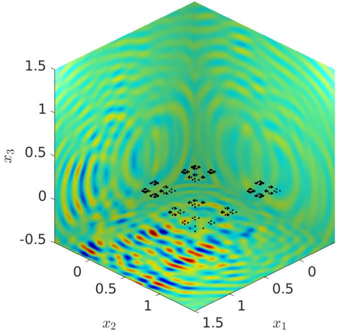


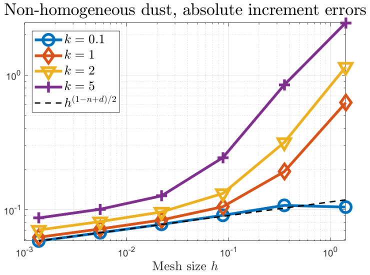


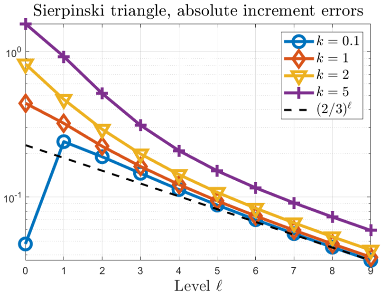
▲ Non-homogeneous disjoint IFS attractor
with $M = 4$, $\rho_{1,2,3} = \frac{1}{4}$, $\rho_4 = \frac{1}{2}$, $d = \frac{\log 3}{\log 2}$

$n = 3$, flat screen: non-homogeneous dust & Sierpinski triangle



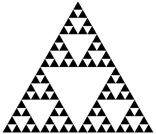






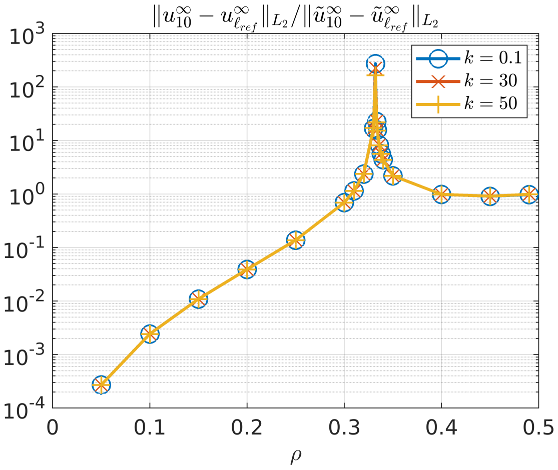
▲ Non-homogeneous disjoint IFS attractor with $M = 4$, $\rho_{1,2,3} = \frac{1}{4}$, $\rho_4 = \frac{1}{2}$, $d = \frac{\log 3}{\log 2}$

◀ Sierpinski triangle is not disjoint: does not satisfy BEM convergence theory assumptions.



Comparison against "prefractal-BEM" for Cantor sets

Prefractal-BEM solution $\tilde{u}$ computed on Lipschitz prefractal approximations of Γ as in (Chandler-Wilde, Hewett, Moiola, Besson, 2021)



Compare far-fields on circle "at infinity"

◀ Ratio between Hausdorff-BEM and prefractal-BEM errors.

Same number of DOFs
($\approx$ computational effort).

$\rho < 0.3$: Hausdorff-BEM is far more accurate

$\rho \approx 1/3$: Lebesgue-BEM has strange "enhanced accuracy"

$\rho > 0.4$: the methods are comparable

Results are independent of wavenumber k.

Summary and outlook

Scattering of time-harmonic acoustic waves by sound-soft obstacle Γ:

Γ compact:	BVP is well-posed, equivalent to IE
Γ d-set:	IE in Hausdorff measure, convergence of piecewise-constant Galerkin
Γ disjoint IFS:	concrete recipe for Galerkin space & quadrature, convergence rates

Fractal IFS Γ is not approximated. Only function (space) and integral are approximated.

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Open questions and ongoing work:

- Solution regularity theory ($\phi \in H_\Gamma^{-\frac{n-d}{2}-\epsilon}$), singularity structure
- Non-disjoint attractors , $d = n$ 
- Fast implementation, compression
- Maxwell equations? Other PDEs? (Laplace & reaction–diffusion already covered)
- Volume integral equation, penetrable materials
- IFSs with non-similar contractions, . . .

A. Caetano, S.N. Chandler-Wilde, X. Claeys, A. Gibbs, D.P. Hewett, A. Moiola,
Integral equation methods for acoustic scattering by fractals arXiv:2309.02184
julia code @: `https://github.com/AndrewGibbs/IFSintegrals`

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