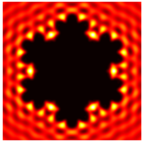
Fractals in pure and applied sciences — Roma — 15–17 March 2023

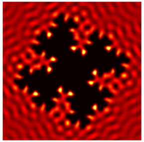
A Hausdorff-measure boundary element method for acoustic scattering by fractal screens

Andrea Moiola

http://matematica.unipv.it/moiola/



Università di Pavia
Department of Mathematics "Felice Casorati"



A. Caetano (Aveiro), S.N. Chandler-Wilde (Reading), A. Gibbs (UCL), D.P. Hewett (UCL)

arXiv:2212.06594

Acoustic wave scattering by a planar screen

Acoustic waves in free space ($\mathbb{R}^{n+1}$) are governed by the wave equation $\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

In time-harmonic regime, assume $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}kt}\}$ and look for u.
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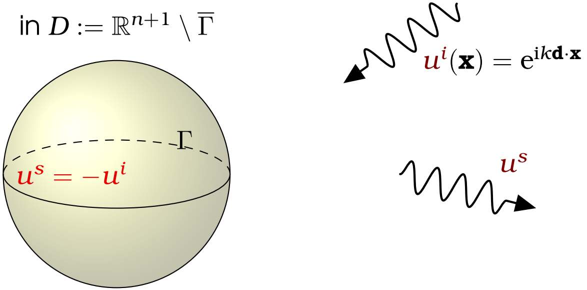
$$\Delta u^s + k^2 u^s = 0$$

in $D := \mathbb{R}^{n+1} \setminus \overline{\Gamma}$

$u^{tot} = u^i + u^s$



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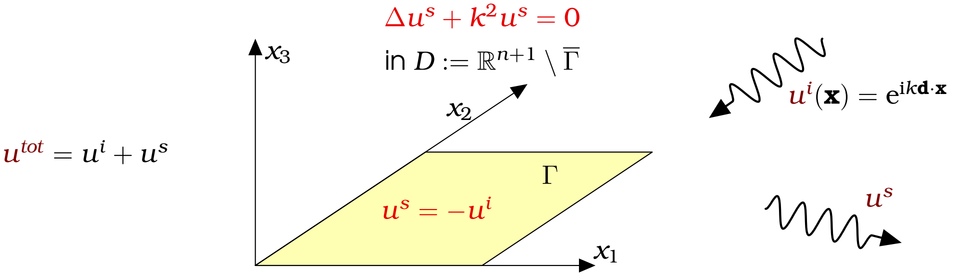

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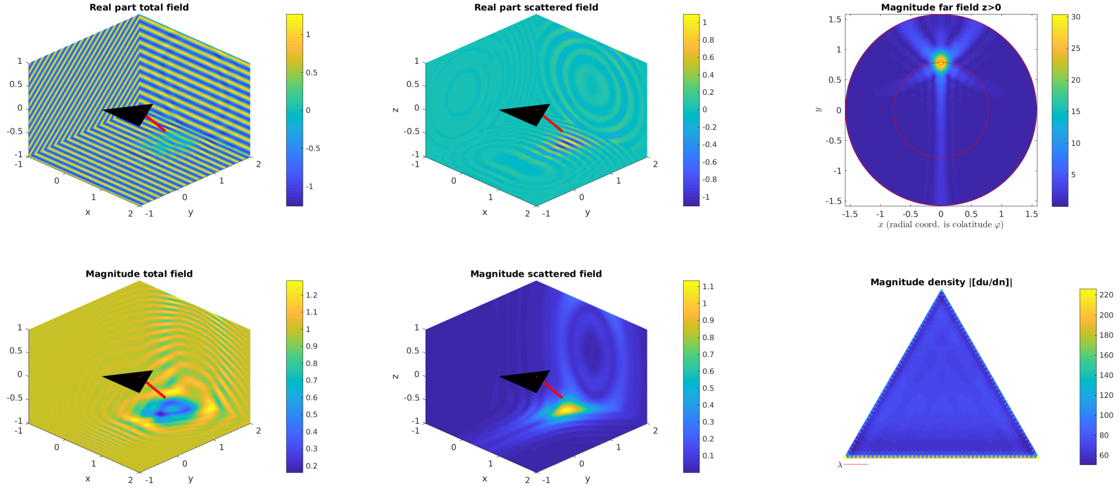
Planar screen obstacle: Γ bounded subset of $\Gamma_\infty := \{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\} \cong \mathbb{R}^n$, $n = 1, 2$.

Scattering by Lipschitz and rough screens

Incident field is plane wave $u^i(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{d}\cdot\mathbf{x}}$, $|\mathbf{d}| = 1$.



Classical problem when Γ is open and Lipschitz.

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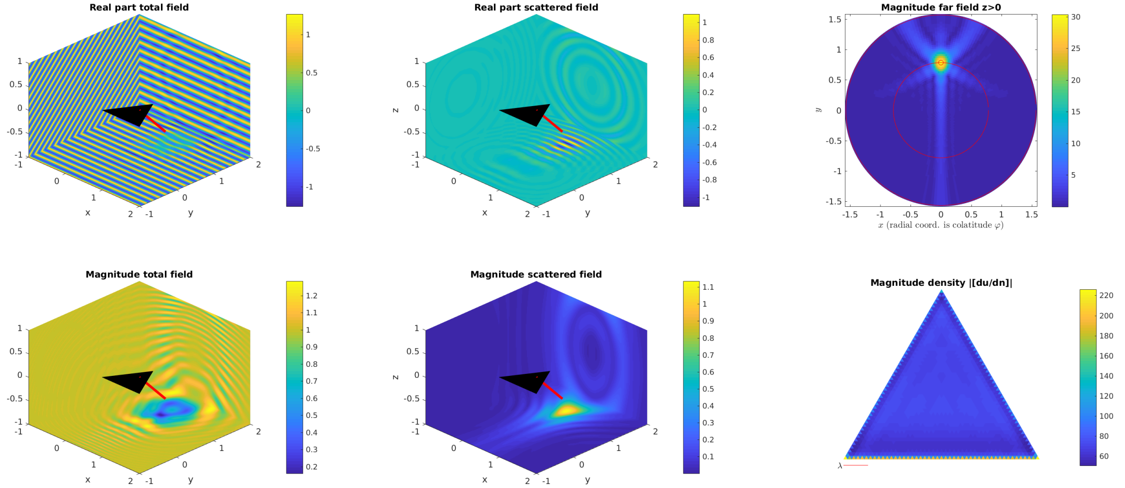
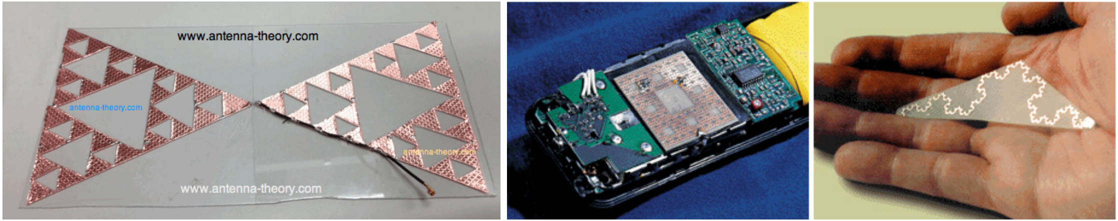


Classical problem when Γ is open and Lipschitz.

What happens for rougher than Lipschitz, e.g. fractal, Γ?

Waves and fractals: applications

Wideband fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

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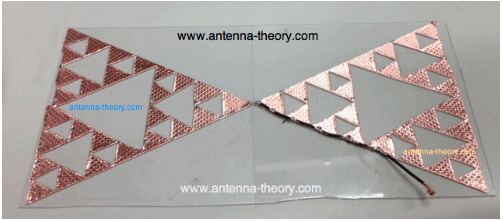

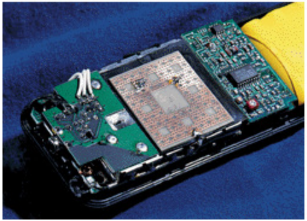
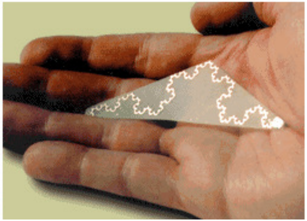

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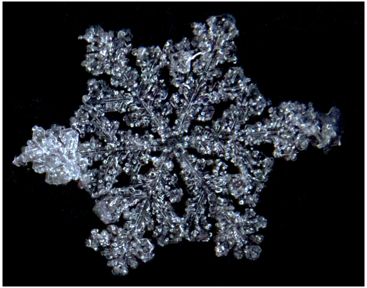
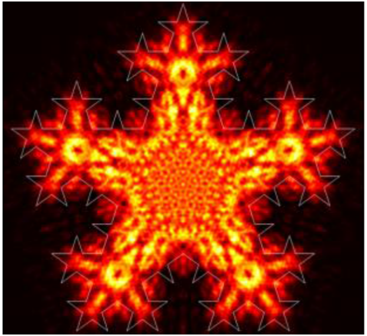


Scattering by ice crystals
in atmospheric physics
(C. Westbrook)

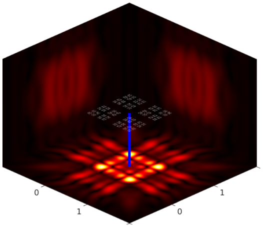
Fractal apertures
in laser optics
(J. Christian)

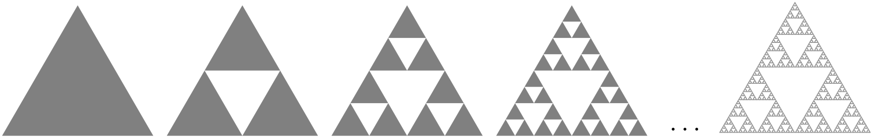


Scattering by fractal screens

Plenty of mathematical challenges:

- How to formulate well-posed BVPs?
  What is the right function space setting?
  How to impose BCs?
  How to write BVP as integral equation?
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?
- ...





Tools developed here (hopefully!) relevant to (numerical) analysis of
other IEs, ΨDOs, BVPs, integration on rough/complicated/fractal domains.

Our main contributions

BVPs, formulations, function spaces

- SCW, DH, IEOT, 2015
  Wavenumber-explicit continuity & coercivity est. in acoustic scattering by planar scr.
- SCW, DH, AM, IEOT, 2017
  Sobolev spaces on non-Lipschitz subsets of $\mathbb{R}^n$ with application to BIEs on fractal scr.
- SCW, DH, SIAM J. Math. Anal., 2018
  Well-posed PDE and integral equation formulations for scattering by fractal screens,
- AC, DH, AM, JFA 2021
  Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets

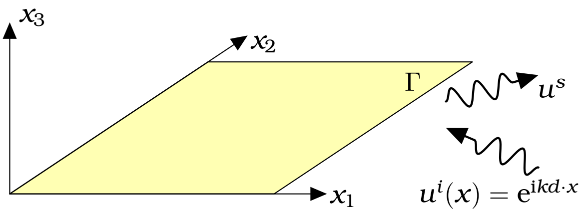
Numerical methods

- SCW, DH, AM, J.Besson, Numer. Math., 2021
  Boundary element methods for acoustic scattering by fractal screens
- J.Bannister, AG, DH, M3AS 2022
  Acoustic scattering by impedance screens/cracks with fractal boundary: well-posedness analysis and boundary element approximation
- AG, DH, AM, Numer. Algorithms, 2022
  Numerical quadrature for singular integrals on fractals
- AC, SCW, AG, DH, AM, arXiv:2212.06594, 2022
  A Hausdorff-measure BEM for acoustic scattering by fractal screens

A crash course in BIEs and BEM (boundary element method)

BVP:

$$\begin{cases} \Delta u^s + k^2 u^s = 0 & D := \mathbb{R}^{n+1} \setminus \Gamma \\ \partial_r u^s - \mathrm{i}k u^s = o(r^{-\frac{n}{2}}) & r = |\mathbf{x}| \to \infty \\ u^s = -u^i & \Gamma \subset \Gamma_\infty \cong \mathbb{R}^n \end{cases}$$



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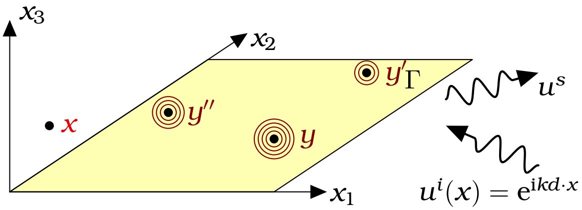

- **Represent** scattered field in D e.g. as $u^s(x) = \mathcal{S}\phi(x) = -\int_\Gamma \Phi(x,y)\phi(y)\,\mathrm{d}s(y),\ x \in D$
  $\mathcal{S}$ is a "layer potential" (a superposition of point sources on Γ),
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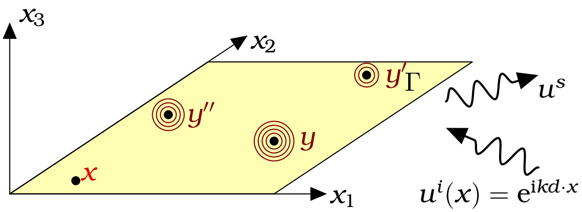

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- Derive a **BIE** for ϕ, e.g. $S\phi = g$, $\qquad S : V \to V^*, \quad V = H_\Gamma^{-1/2}$
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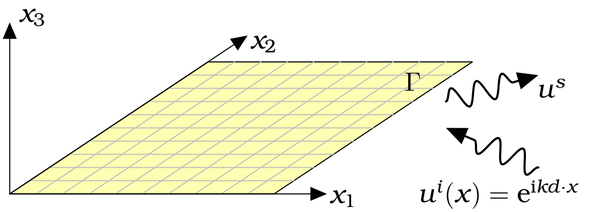

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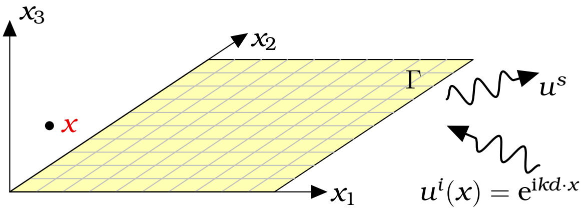

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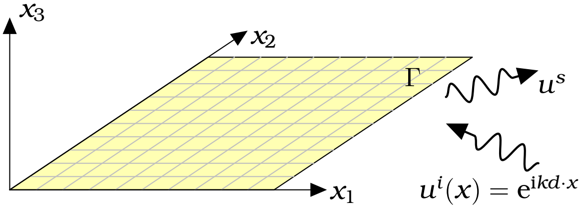

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- **Evaluate** $u^s_N(x) = (\mathcal{S}\phi_N)(x) \approx u^s(x)$ for $x \in D$

Theorem (SCW, DH 2018): For any compact $\Gamma \subset \Gamma_\infty$, BVP is well-posed & equivalent to BIE

Two ways to apply BEM to fractal Γ

1. (Chandler-Wilde, Hewett, Moiola, Besson, 2021)

2. (Caetano, Chandler-Wilde, Gibbs, Hewett, Moiola, arXiv:2212.06594)

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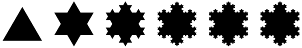
Approximate Γ with Lipschitz "prefractal" Γ_j and apply conventional BEM on each Γ_j

open $\Gamma_j \subset \Gamma_{j+1}$



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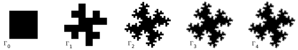

- "Non-conforming", since typically $V_N \not\subset V = H_\Gamma^{-1/2}$
- BVP and BEM convergence from Mosco convergence of spaces
- No convergence rates
- Requires "thickened prefractals"
- Can use any BEM implementation

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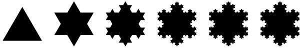

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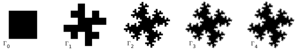

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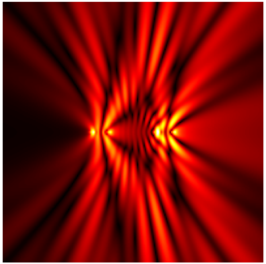
2 (CAETANO, CHANDLER-WILDE, GIBBS, HEWETT, MOIOLA, arXiv:2212.06594)

- Directly discretise Γ, integration wrt Hausdorff measure
- Conforming method $V_N \subset V = H_\Gamma^{-1/2}$
- Easy convergence from Céa lemma + rates
- Require special quadrature formulas

Rest of this talk!

What do we do?

- d-sets:
  function spaces, trace operators
  integral operators, BIEs, variational forms
  Galerkin method, piecewise-constant BEM
  Theorem: BEM convergence
- Disjoint IFS attractors:
  IFS, tree structure, wavelets
  piecewise-constant BEM space
  Theorem: BEM convergence rates



- Numerical results:
  Cantor sets, dusts, non-homogeneous sets, Sierpinski triangle
- Numerical integration on IFS attractors:
  barycentre rule for smooth integrand
  self-similarity for homogeneous singular integrals
  rule for Helmholtz kernel
  numerical examples
  comparison with chaos game

Part I

BIE and BEM on d-sets

d-sets and function spaces

A compact set $\Gamma \subset \mathbb{R}^n$ is a d-set if $c_1 r^d \le \mathcal{H}^d(\Gamma \cap B_r(x)) \le c_2 r^d$ $\quad x \in \Gamma,\ 0 < r \le 1$

"Uniformly locally d-dimensional sets". FALCONER, TRIEBEL, JONSSON&WALLIN, . . .

E.g.: Cantor sets/dusts, Sierpinski, Menger, snowflakes, . . . Closure of Lipschitz is n-set

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Classical function spaces, "extrinsic" on $\mathbb{R}^n$ & "intrinsic" on Γ:

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^n) :\ \|u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \mathrm{d}\xi < \infty \right\}$$

$$\mathbb{L}_2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C} :\ \|f\|^2_{\mathbb{L}_2(\Gamma)} = \int_{\Gamma} |f(x)|^2 \mathrm{d}\mathcal{H}^d(x) < \infty \right\}$$

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Trace operator: define $\mathrm{tr}_\Gamma \varphi = \varphi|_\Gamma$ for $\varphi \in C^\infty(\mathbb{R}^n)$.
For $s > \frac{n-d}{2}$, it extends to $\mathrm{tr}_\Gamma : H^s(\mathbb{R}^n) \to \mathbb{L}_2(\Gamma)$ (continuous linear op. with dense image)

d-sets and function spaces

A compact set $\Gamma \subset \mathbb{R}^n$ is a d-set if $c_1 r^d \le \mathcal{H}^d(\Gamma \cap B_r(x)) \le c_2 r^d \quad x \in \Gamma,\ 0 < r \le 1$

"Uniformly locally d-dimensional sets". FALCONER, TRIEBEL, JONSSON&WALLIN, ...
E.g.: Cantor sets/dusts, Sierpinski, Menger, snowflakes, ... Closure of Lipschitz is n-set

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Classical function spaces, "extrinsic" on $\mathbb{R}^n$ & "intrinsic" on Γ:

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Single-layer operator on d-sets

From now on, assume that scatterer Γ is a d-set with $n-1 < d \leq n$.
Γ produces scattered wave $u^s \neq 0$. $(u^s = 0$ if $d \leq n-1)$

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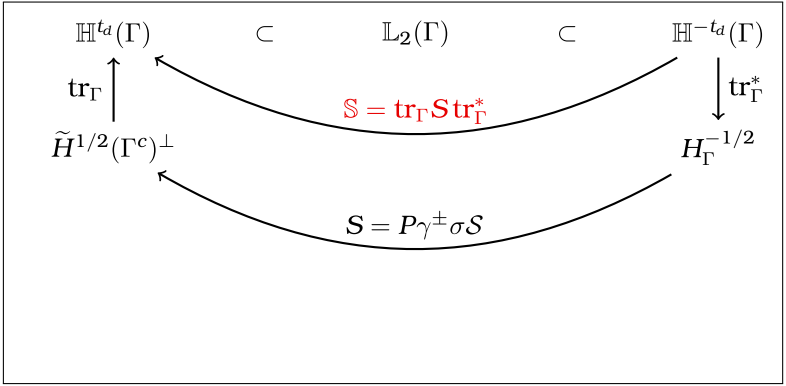

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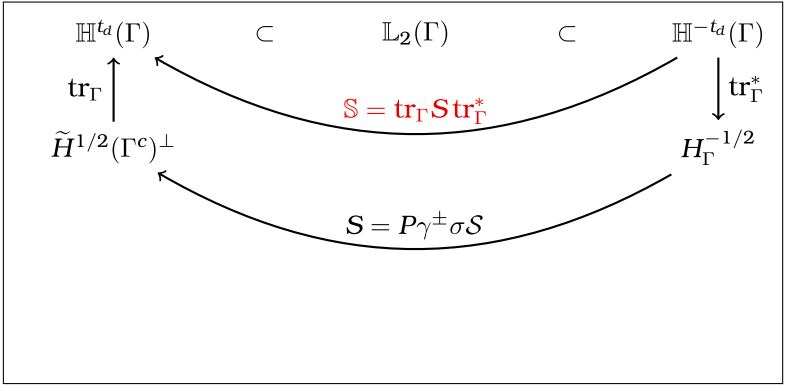

Theorem

$\mathbb{S}$ is integral operator in Hausdorff measure:
$\forall\Psi\in L_\infty(\Gamma)$

$$\mathbb{S}\Psi(x)=\int_\Gamma\Phi(x,y)\Psi(y)\,\mathrm{d}\mathcal{H}^d(y)$$

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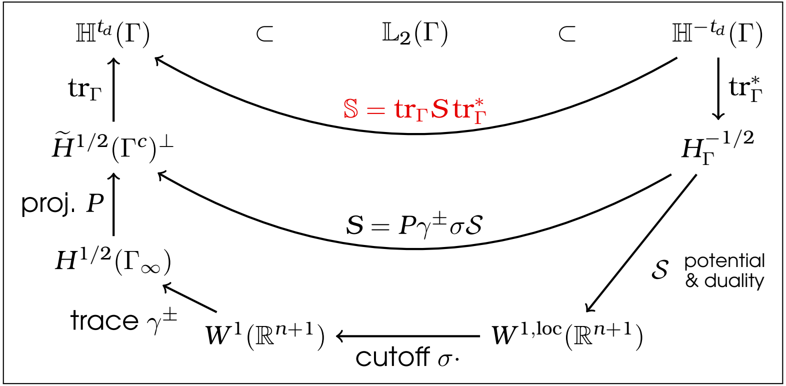

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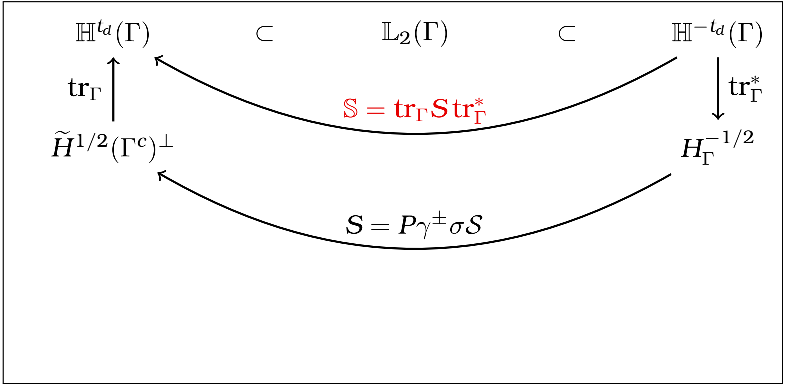

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$\mathbb{S} : \mathbb{H}^{t-t_d}(\Gamma) \to \mathbb{H}^{t+t_d}(\Gamma)$

Continuous for $|t| < t_d$

Coercive and invertible for $t = 0$

Conjecture: $\mathbb{S}$ invertible for $|t| < t_d$ (true for Lipschitz Γ, $d = n$)

Conjecture would imply regularity for scattering BIE: $\phi \in H_\Gamma^{-\frac{n-d}{2}-\epsilon}$

Variational problems and Galerkin method on d-sets

Two equivalent variational problems. Datum: $g \in \widetilde{H}^{1/2}(\Gamma^c)^\perp$ (trace of u^i).
"Extrinsic form":

$$\text{find } \phi \in H_\Gamma^{-1/2}, \qquad \langle S\phi, \psi\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)} = -\langle g, \psi\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)} \qquad \forall \psi \in H_\Gamma^{-1/2}$$

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Theorem (Chandler-Wilde, Hewett 2015): sesquilinear forms are continuous & coercive

$\Rightarrow$ Both variational problems are well-posed. $\quad \phi = \mathrm{tr}_\Gamma^* \widetilde{\phi}$

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$$\text{find } \phi_N \in V_N, \qquad \langle S\phi_N, \psi_N\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)} = -\langle g, \psi_N\rangle_{H^{1/2}(\Gamma_\infty)\times H^{-1/2}(\Gamma_\infty)} \qquad \forall \psi_N \in V_N$$

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are well-posed. If $d < n$, $\mathbb{V}_N \subset \mathbb{L}_2(\Gamma)$ is possible, $H_\Gamma^0 = L_2(\Gamma) = \{0\}$

Piecewise-constant BEM on d-sets

Finding $\widetilde{\phi}_N = \sum_{j=1}^{n} c_j f^j \in \mathbb{V}_N$, $\quad \langle \mathbb{S}\widetilde{\phi}_N, \widetilde{\psi}_N \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} = -\langle \mathrm{tr}_\Gamma g, \widetilde{\psi}_N \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} \quad \forall \widetilde{\psi}_N \in \mathbb{V}_N$

where $\{f^j\}_{j=1}^N$ is a basis of $\mathbb{V}_N$, is equivalent to solving the $N \times N$ linear system

$$A\vec{c} = \vec{b}, \qquad A_{ij} := \langle \mathbb{S} f^j, f^i \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)}, \qquad b_i := -\langle \mathrm{tr}_\Gamma g, f^i \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)}, \qquad i,j = 1,\ldots,N.$$

Can choose $\mathbb{V}_N \subset \mathbb{L}_2(\Gamma) \overset{\text{dense}}{\subset} \mathbb{H}^{-t_d}(\Gamma)$. Need to compute integrals wrt $\mathcal{H}^d$!

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with $\mathcal{H}^d$-measurable elements T_j, $\quad \mathcal{H}^d(T_j) > 0$, $\quad \mathcal{H}^d(T_j \cap T_i) = 0$ for $j \neq i$.

$\mathbb{L}_2(\Gamma)$-orthonormal basis: $\quad f^j(x) = (\mathcal{H}^d(T_j))^{-1/2}$ for $x \in T_j$, $\; f^j(x) = 0$ otherwise.

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Theorem: BEM convergence for d-sets

For a sequence $(\mathbb{V}_N)_{N\in\mathbb{N}}$ of discrete spaces, $\quad \widetilde{\phi}_N \to \widetilde{\phi} \quad$ if $h_N := \max_{j=1,\ldots,N} \mathrm{diam}(T_j) \to 0$.

How to get convergence rates? We need stronger assumptions on Γ.

Part II

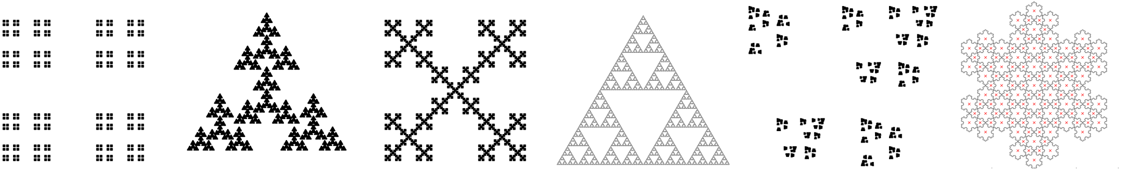
BEM on IFS attractors

Iterated function systems (IFS)

IFS is a family of M contracting similarities:

$$s_m : \mathbb{R}^n \to \mathbb{R}^n, \qquad |s_m(x) - s_m(y)| = \rho_m |x - y|, \qquad 0 < \rho_m < 1, \qquad m = 1, \dots, M.$$

There exists a unique non-empty compact Γ with $\Gamma = s(\Gamma)$, where $s(E) := \bigcup_{m=1}^{M} s_m(E)$.



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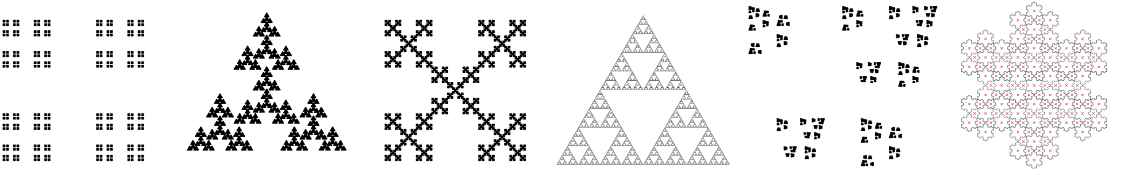
Assume open set condition (OSC): $\exists O \subset \mathbb{R}^n$ open, $s(O) \subset O$, $s_m(O) \cap s_{m'}(O) = \emptyset \ \forall m \neq m'$.
Then Γ is d-set, $\sum_{m=1}^{M} \rho_m^d = 1$.



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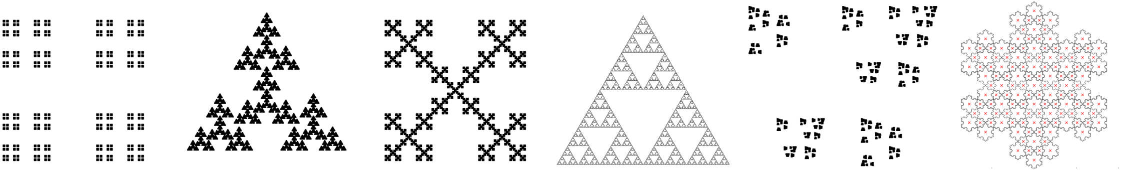
IFS is homogeneous if $\rho_m = \rho \; \forall m$ (then $d = \frac{\log M}{\log 1/\rho}$).

Γ is disjoint if $\Gamma_m := s_m(\Gamma)$ are all disjoint. (Falconer, Hutchinson, Triebel,...)
Disjoint implies OSC and $d < n$.



IFS tree structure and wavelets

Disjoint IFS attractors have natural tree structure:

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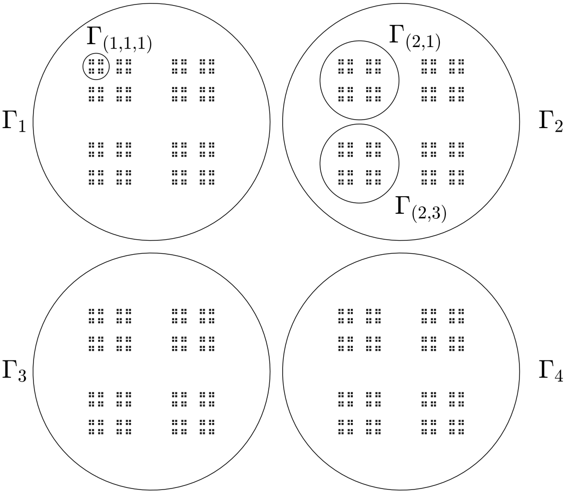

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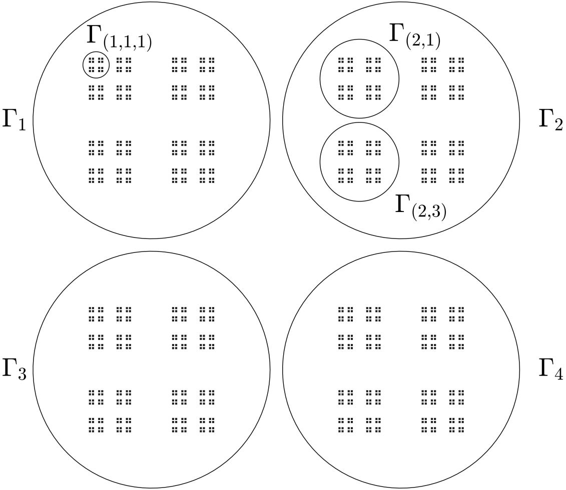
Disjoint IFS attractors have natural tree structure:

$$\Gamma_0 := \Gamma, \qquad \Gamma_{\mathbf{m}} := s_{\mathbf{m}}(\Gamma), \qquad s_{\mathbf{m}} := s_{m_1} \circ \ldots \circ s_{m_\ell}, \quad \mathbf{m} = (m_1, \ldots, m_\ell) \in \{1, \ldots, M\}^\ell, \quad \ell \in \mathbb{N}$$



Characteristic functions:

$$\chi_{\mathbf{m}}(x) := \begin{cases} 1 & x \in \Gamma_{\mathbf{m}} \\ 0 & \text{otherwise} \end{cases}$$

Linear combinations give hierarchical orthonormal wavelet basis of $\mathbb{L}_2(\Gamma)$.

Collecting $\Gamma_{\mathbf{m}}$s according to diameter, wavelet basis gives characterisation of $\mathbb{H}^t(\Gamma)$ and its norm.
(Jonsson 1998)

$\{\mathbb{H}^t(\Gamma)\}_{|t|<1}$ & $\{H^s_\Gamma\}_{-(n-d)/2-1<s<-(n-d)/2}$
are interpolation scales

Piecewise-constant BEM space on IFS attractor

We exploit IFS tree structure to construct BEM space and basis: $0 < h < \mathrm{diam}(\Gamma)$

$$\mathbb{V}_N = \mathrm{span}\Big\{\chi_{\mathbf{m}},\ \mathbf{m} \in \{1,\ldots,M\}^\ell, \ell \in \mathbb{N},\ \ \mathrm{diam}(\Gamma_{\mathbf{m}}) \leq h,\ \ \mathrm{diam}(\Gamma_{(m_1,\ldots,m_{\ell-1})}) > h\Big\} \subset \mathbb{L}_2(\Gamma)$$

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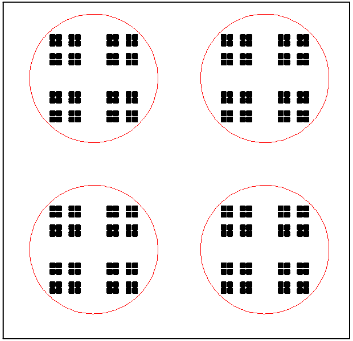
Each $T_j = \Gamma_{\mathbf{m}}$ is a copy of Γ under similarity $s_{\mathbf{m}}$, with $\mathrm{diam}(T_j) \le h$.

$\mathrm{diam}(\Gamma) = \sqrt{2}$, $M = 4$



$\rho = \frac{1}{3}$, $h = 0.5$, $N = 4$

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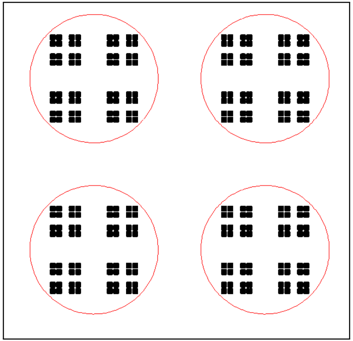

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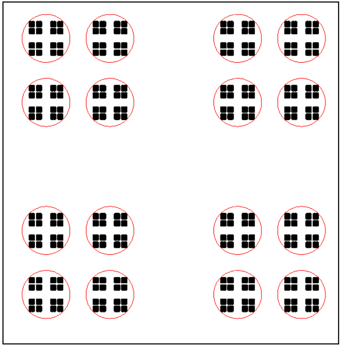

$\rho = \frac{1}{3}$, $h = 0.2$, $N = 16$

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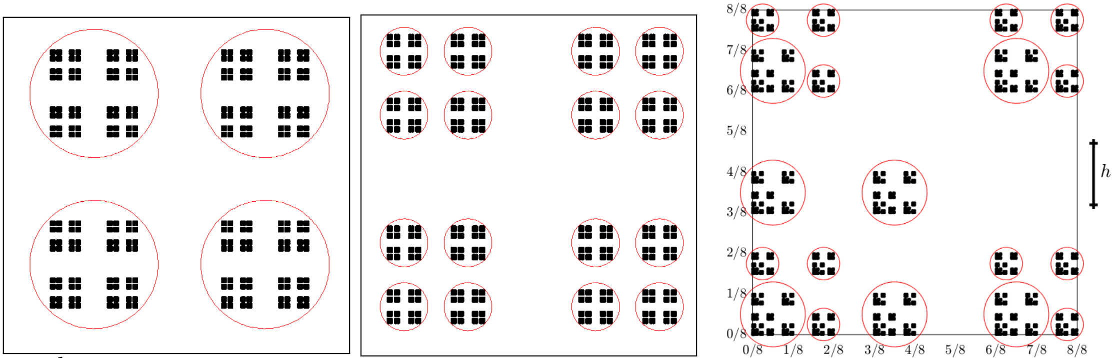

$\rho = \frac{1}{3}$, $h = 0.5$, $N = 4$ $\quad$ $\rho = \frac{1}{3}$, $h = 0.2$, $N = 16$ $\quad$ $\rho_1 = \frac{1}{2}$, $\rho_{2:4} = \frac{1}{4}$, $h = 0.2$, $N = 19$

Piecewise-constant BEM convergence for disjoint IFS attractors

Using coercivity, Céa, relation BEM space/wavelets, coefficient decay in $\mathbb{H}^t(\Gamma)$:

Theorem (CCGHM 2022)

Γ disjoint IFS attractor. Assume BIE solution $\phi \in H^s_\Gamma$ for some $-\frac{1}{2} < s < -\frac{n-d}{2}$. Then

$$\|\widetilde{\phi} - \widetilde{\phi}_N\|_{\mathbb{H}^{-\frac{1}{2}+\frac{n-d}{2}}(\Gamma)} = \|\phi - \phi_N\|_{H_\Gamma^{-\frac{1}{2}}} \leq c h^{s+\frac{1}{2}} \|\phi\|_{H^s_\Gamma}$$

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- h^{2s+1} super-convergence of linear functionals, e.g.: point value $u^s(x)$ and far-field
- Regularity assumption on ϕ implied by previous conjecture on $\mathbb{S}$ $\quad H_\Gamma^{-\frac{n-d}{2}} = \{0\}$
- For homogeneous IFS, if conjecture is valid, rates are

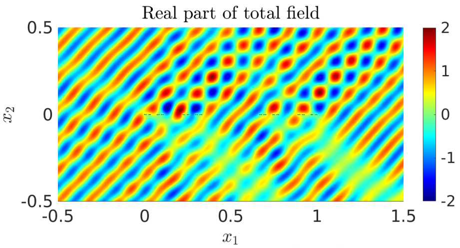
$$M^{-\ell/2} \quad \text{for } n = 1, \qquad (\rho M)^{-\ell/2} \quad \text{for } n = 2$$

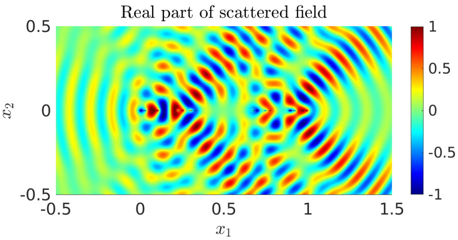
  with ℓ the "level" of the BEM space
- In the limit $d \nearrow n$, we recover classical results for Lipschitz screens
- Inverse estimates in $\mathbb{V}_N$: bound $H^{s_1}_\Gamma$ error norm $-1/2 < s_1 < s$ and condition number
- Can control "fully discrete error" taking into account numerical integration

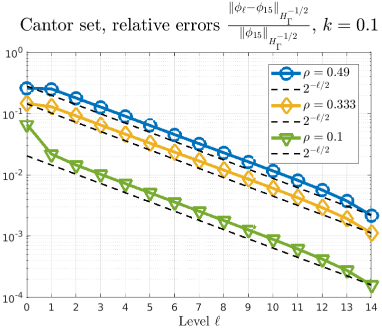
Part III

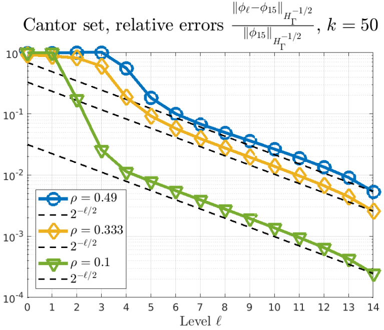
Numerical results

2D scattering problem: Cantor set $\Gamma \subset \mathbb{R}$





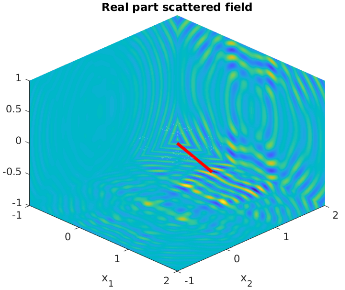


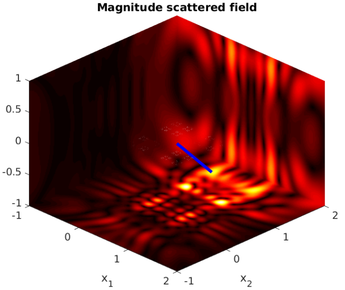


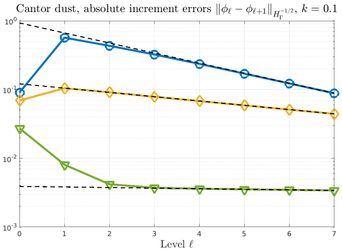
Rate $2^{-\ell/2}$ in $H_\Gamma^{-1/2}$ norm as expected, independent of ρ. $u^i(x) = \mathrm{e}^{\mathrm{i}k\theta\cdot x}$

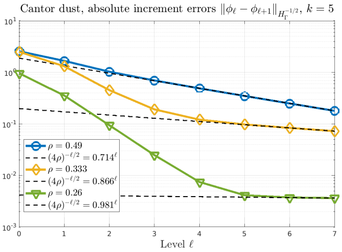
Similar plots (with double rate $2^{-\ell}$) for near-field $u^s(x)$ and far-field.

3D scattering problem: Cantor dust $\Gamma \subset \mathbb{R}^2$



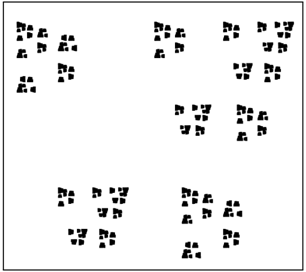


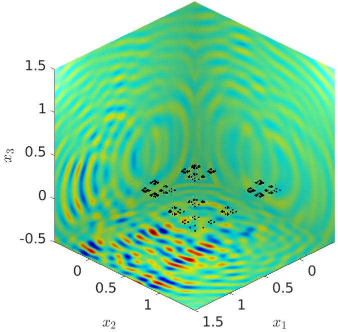


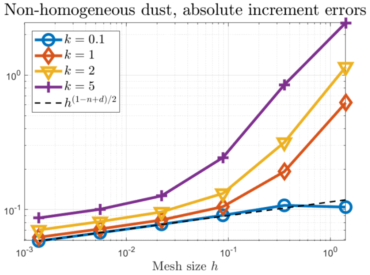


ρ-dependent rate $(4\rho)^{-\ell/2}$ in $H_\Gamma^{-1/2}$ norm as expected.
Double rates $(4\rho)^{-\ell}$ for near-field and far-field.

Non-homogeneous dust and Sierpinski triangle in $\mathbb{R}^2$

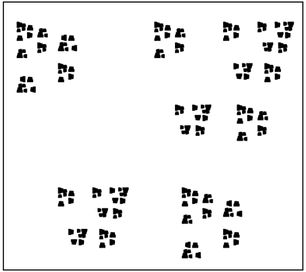


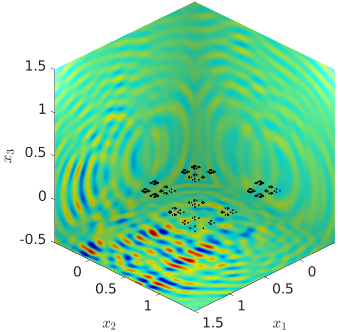


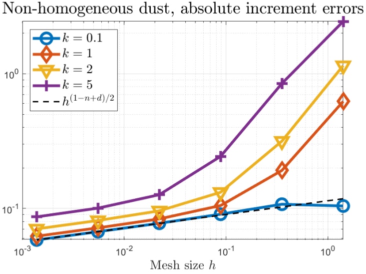


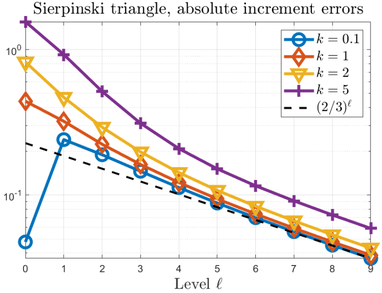
▲ Non-homogeneous disjoint IFS attractor with $M = 4, \quad \rho_{1,2,3} = \frac{1}{4}, \quad \rho_4 = \frac{1}{2}, \quad d = \frac{\log 3}{\log 2}$

Non-homogeneous dust and Sierpinski triangle in $\mathbb{R}^2$









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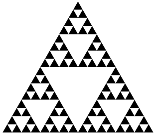
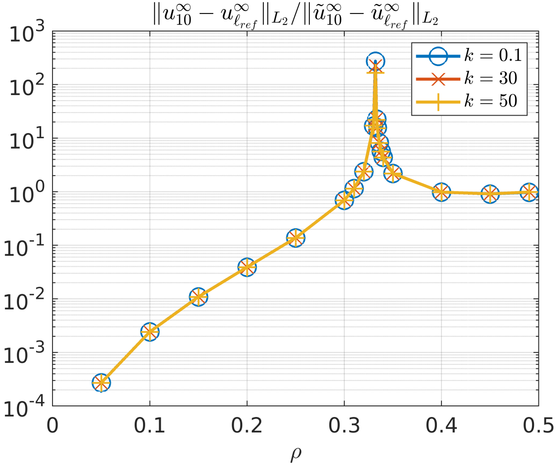
◀ Sierpinski triangle is not disjoint: does not satisfy BEM convergence theory assumptions.



Comparison against "prefractal-BEM" for Cantor sets in $\mathbb{R}$

Prefractal-BEM solution $\tilde{u}$ computed on Lipschitz prefractal approximations of Γ as in (Chandler-Wilde, Hewett, Moiola, Besson, 2021)



Compare far-fields on circle "at infinity"

◀ Ratio between Hausdorff-BEM and prefractal-BEM errors.

Same number of DOFs ($\approx$ computational effort).

$\rho < 0.3$: Hausdorff-BEM is far more accurate

$\rho \approx 1/3$: Lebesgue-BEM has strange "enhanced accuracy"

$\rho > 0.4$: the methods are comparable

Results are independent of wavenumber k.

Part IV

Numerical quadrature

Numerical integration on IFS attractors

Each element of the Galerkin matrix is double singular integral wrt Hausdorff measure:

$$A_{jj'} = \langle \mathbb{S}\chi_{\mathbf{m}'}, \chi_{\mathbf{m}} \rangle_{\mathbb{H}^{t_d}(\Gamma)\times\mathbb{H}^{-t_d}(\Gamma)} = \int_\Gamma \int_\Gamma \Phi(x,y)\chi_{\mathbf{m}'}(x)\chi_{\mathbf{m}}(y)\mathrm{d}\mathcal{H}^d(x)\mathrm{d}\mathcal{H}^d(y)$$

$$= \int_{\Gamma_{\mathbf{m}}} \int_{\Gamma_{\mathbf{m}'}} \Phi(x,y)\mathrm{d}\mathcal{H}^d(x)\mathrm{d}\mathcal{H}^d(y)$$

$$\Phi(x,y) = \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} \quad \text{if } n=2$$

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Numerical quadrature for singular integrals on fractals

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Consider Hausdorff and more general "invariant measures": (HUTCHINSON 1981)

Given IFS $s_1,\dots,s_M$ and $p_1,\dots,p_M \in (0,1)$, $\sum_{m=1}^M p_m = 1$,

$\exists!$ Borel μ s.t. $\mu(A) = \sum_{m=1}^M p_m \mu(s_m^{-1}(A))$, $\mathrm{supp}(\mu) = \Gamma$ $\qquad$ ($p_m = \rho_m^d$ if $\mu = \mathcal{H}^d$)

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3 quadrature rules:

- Barycentre rule for "smooth" (C^1 and C^2) integrands
- Self-similar rule for homogeneous singular integrands $\quad |x-y|^{-t}$ or $\log|x-y|$
- Singularity-subtraction rule for Helmholtz fundamental solution $\quad \Phi(x,y) = \frac{1}{4\pi|x-y|} + \mathcal{R}$

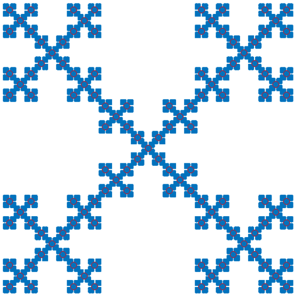
Each $\Gamma_{\mathbf{m}}$ is similar copy of Γ: for simplicity we just consider integrals over Γ.

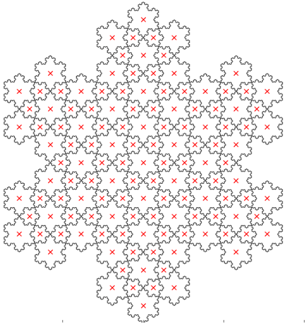
Barycentre rule for smooth integrals

As before, partition Γ in $\Gamma_{\mathbf{m}} = s_{\mathbf{m}}(\Gamma)$ with $\mathrm{diam}(\Gamma_{\mathbf{m}}) \approx h_Q$.

Extend classical midpoint rule:
Approximate $f|_{\Gamma_{\mathbf{m}}}$ with $f(\mathbf{x}_{\mathbf{m}})$, where $x_{\mathbf{m}}$ is barycentre of $\Gamma_{\mathbf{m}}$

$$\int_\Gamma f(x)\mathrm{d}\mu(x) = \sum_{\mathbf{m}} \int_{\Gamma_{\mathbf{m}}} f(x)\mathrm{d}\mu(x) \approx \sum_{\mathbf{m}} \mu(\Gamma_{\mathbf{m}}) f(\mathbf{x}_{\mathbf{m}})$$





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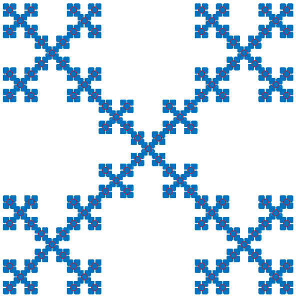
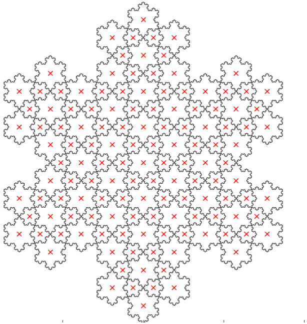


Barycentre and weights are easily computed:

$$\mu(\Gamma_{\mathbf{m}}) = p_{m_1} \cdots p_{m_\ell} \mu(\Gamma),$$

$$x_{\mathbf{m}} = \frac{\int_{\Gamma_{\mathbf{m}}} x\mathrm{d}\mu(x)}{\mu(\Gamma_{\mathbf{m}})} = s_{m_1} \circ \cdots \circ s_{m_\ell} \left(\left[I - \sum_{m=1}^{M} p_m \rho_m A_m \right]^{-1} \sum_{m=1}^{M} p_m \delta_m \right)$$

where $\mathbf{m} = (m_1, \ldots, m_\ell) \in (1, \ldots, M)^\ell$, $\qquad s_m(x) = \rho_m A_m x + \delta_m$



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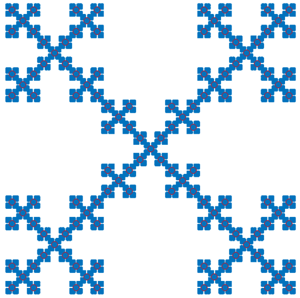

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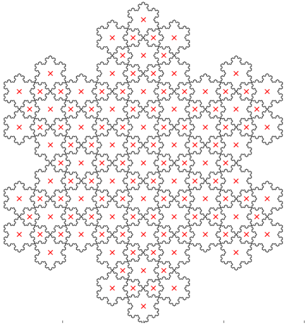


$$\text{Error} \leq \frac{n}{2} h_Q^2 \, \mu(\Gamma) \, |f|_{C^2(\bigcup_{\mathbf{m}} \mathrm{Hull}(\Gamma_{\mathbf{m}}))}$$

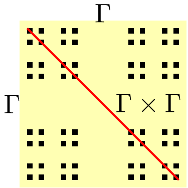
Same story for double integrals.

Quadrature rule for singular homogeneous integrals

Integrability. Γ a compact d-set, $y \in \Gamma$:

$$\int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d, \qquad I^t_{\Gamma,\Gamma} := \int_\Gamma \int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(y) \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d.$$

Singularity of $|x-y|^{-t}$ is localised on the red line.



▲ Example:
Cantor set $\subset \mathbb{R}$
$M = 2$

Quadrature rule for singular homogeneous integrals

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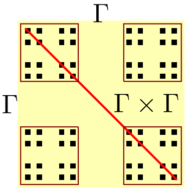
$$\int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d, \qquad I^t_{\Gamma,\Gamma} := \int_\Gamma \int_\Gamma |x-y|^{-t} \mathrm{d}\mathcal{H}^d(y) \mathrm{d}\mathcal{H}^d(x) < \infty \quad \text{iff} \quad t < d.$$



Singularity of $|x-y|^{-t}$ is localised on the red line.

Decompose double integral over $\Gamma \times \Gamma$: $$I^t_{\Gamma,\Gamma} = \sum_{m=1}^{M} \sum_{m'=1}^{M} I^t_{\Gamma_m,\Gamma_{m'}}$$

▲ Example:
Cantor set $\subset \mathbb{R}$
$M = 2$

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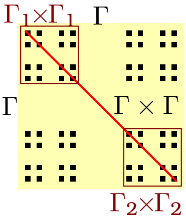

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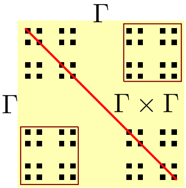

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Can compute $I^t_{\Gamma,\Gamma}$ only in terms of (smooth!) off-diagonal integrals:

$$I^t_{\Gamma,\Gamma} = \frac{1}{1-\sum_{m=1}^M \rho_m^{2d-t}} \sum_{m=1}^M \sum_{\substack{m'=1 \\ m' \neq m}}^M I^t_{\Gamma_m,\Gamma_{m'}}$$

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Compute $I^t_{\Gamma,\Gamma}$ by applying barycentre rule to smooth $I^t_{\Gamma_m,\Gamma_{m'}}$, $m \neq m'$

All this extends to: $\log|x-y|$, invariant measures $\mu \neq \mu'$, single integrals.

Quadrature and BEM

Split Helmholtz fundamental solution as

$$\Phi(x,y) = \begin{cases} \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) = -\frac{1}{2\pi}\log|x-y| + \mathcal{R}(|x-y|) & \text{in } \mathbb{R}^2 \\ \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|} = \frac{1}{4\pi|x-y|} + \mathcal{R}(|x-y|) & \text{in } \mathbb{R}^3 \end{cases} \qquad \mathcal{R} \text{ Lipschitz}$$

Compute the elements of the Galerkin matrix and RHS vector by approximating homogeneous term with self-similar rule and smooth term $\mathcal{R}$ with barycentre rule.

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BEM error bounds taking into account the approximation of the integrals.

h^2 convergence rate is preserved if $h_Q \lesssim h^{1+d}$ ($h_Q \lesssim h^{1+d/2}$ for homogeneous IFS).
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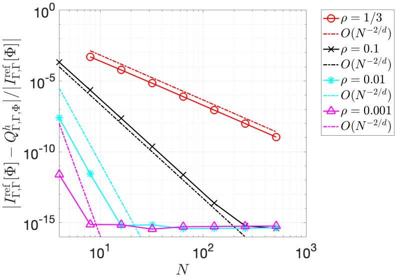
Barycentre rule requires value of $\mathcal{H}^d(\Gamma)$: not known for most fractals $\Gamma \notin \mathbb{R}$!
This is irrelevant for the computation of near-field $u^s(x)$ and far-field in scattering BVP.

Quadrature: numerical examples

Approximation of the integral of the Helmholtz fundamental solution on $\Gamma \times \Gamma$



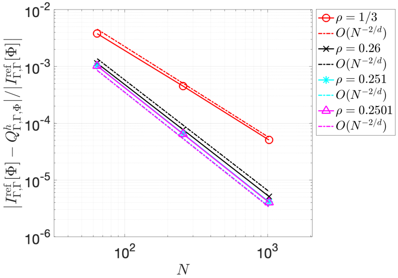
◀ Cantor sets in $\mathbb{R}$

Cantor dusts in $\mathbb{R}^2$ ▶

$k = 5$

Error plotted against
quadrature points

Dashed lines
= theoretical rates



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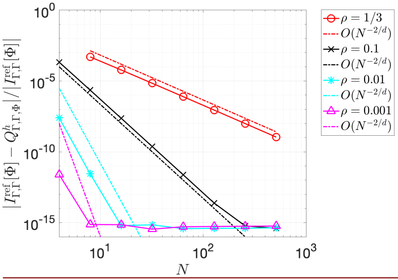

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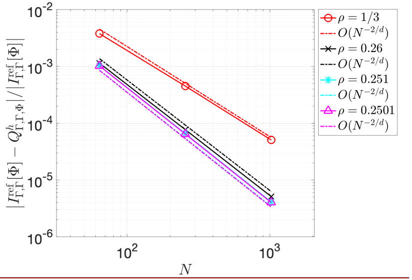

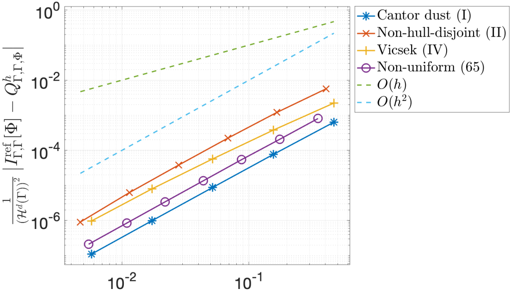

Cantor dust

non “hull-disjoint”

non-disjoint

non-uniform

$k = 2$

Error plotted against $h_{\mathcal{Q}}$

Barycentre rule vs chaos game (Monte Carlo)

Chaos game is alternative quadrature rule: (Forte, Mendivil, Vrscay 1998)

(i) choose $x_0 \in \mathbb{R}^n$
(ii) sequence $\{m_j\}_{j\in\mathbb{N}}$ of i.i.d. random variables in $\{1,\dots,M\}$ with probabilities $\{p_1,\dots,p_M\}$
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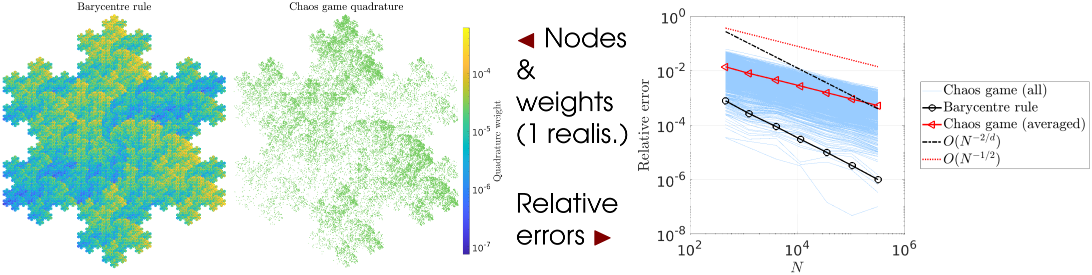
Approximation of $\int_\Gamma f \mathrm{d}\mu$ for $f \in C^\infty$ on $\Gamma =$ Koch snowflake (IFS: $M=7, \rho_{1:6}=\frac{1}{3}, \rho_7=\frac{1}{\sqrt{3}}$)
$\mu =$ invariant measure with non-homogeneous weights p_m. 1000 random realisations.



◀ Nodes & weights (1 realis.)

Relative errors ▶

Summary and outlook

Scattering of time-harmonic acoustic waves by sound-soft planar screen Γ:

Γ compact:	BVP is well-posed, equivalent to BIE
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Open questions and ongoing work:

- Solution regularity theory ($\phi \in H_\Gamma^{-\frac{n-d}{2}-\epsilon}$)
- Non-disjoint attractors  , $d = n$ 
- Non-planar rough scatterers? E.g. $\dim_H(\Gamma) > n-1$, curved screens,...
- Fast BEM implementation
- Maxwell equations? Other PDEs? (Laplace, reaction–diffusion already covered)
- Volume integral equation, penetrable materials, ...

Quadrature: GIBBS, HEWETT, MOIOLA, Numer. Algorithms, 2022

Everything else: CAETANO, CHANDLER-WILDE, GIBBS, HEWETT, MOIOLA, arXiv:2212.06594

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