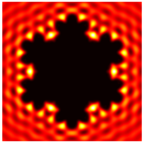
UMI, Pavia, 2–7 September 2019

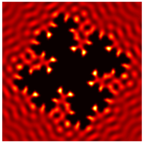
Numerical approximation of acoustic scattering by fractal screens

Andrea Moiola

http://matematica.unipv.it/moiola/



UNIVERSITÀ DI PAVIA
Department of Mathematics
"Felice Casorati"



Joint work with
S.N. Chandler-Wilde (Reading), D.P. Hewett (UCL)
A. Caetano (Aveiro)

Acoustic wave scattering by a planar screen

Time-harmonic (sinusoidal in time) acoustic waves are modelled by the Helmholtz equation $\Delta u + k^2 u = 0$ with wavenumber $k > 0$.

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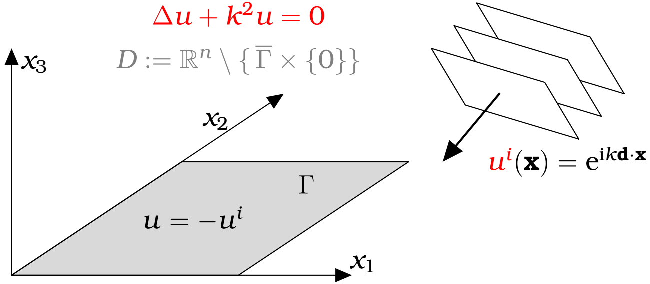
$$D := \mathbb{R}^n \setminus \{\overline{\Gamma} \times \{0\}\}$$



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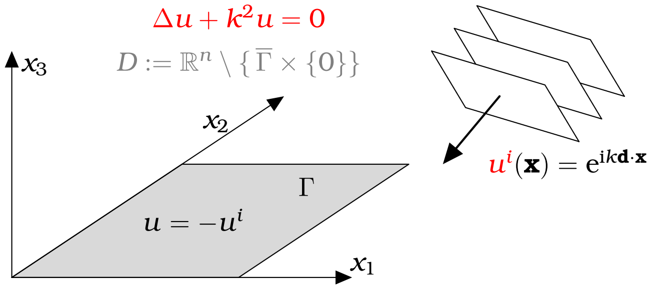

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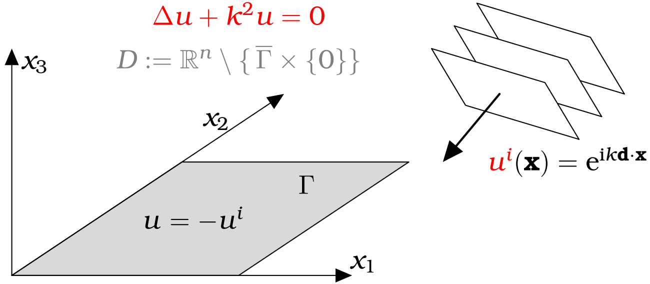

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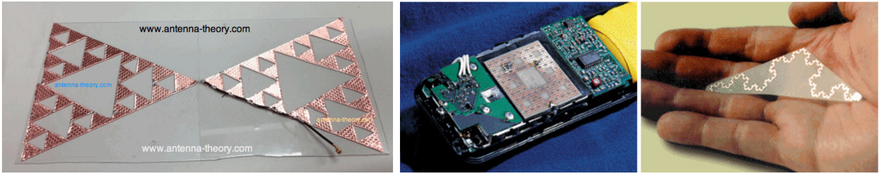
Classical problem when Γ is open and Lipschitz.

What happens for arbitrary (rougher than Lipschitz, e.g. fractal) Γ?

Waves and fractals: applications

Wideband fractal antennas



(Figures from `http://www.antenna-theory.com/antennas/fractal.php`)

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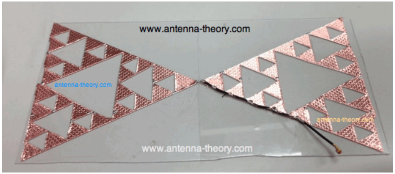

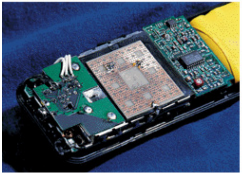

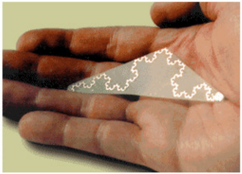

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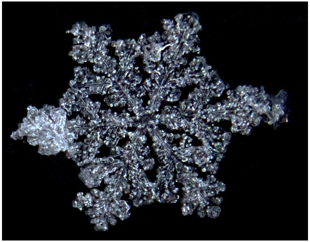
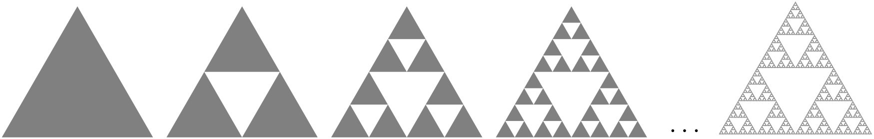
Scattering by ice crystals
in atmospheric physics
e.g. C. Westbrook



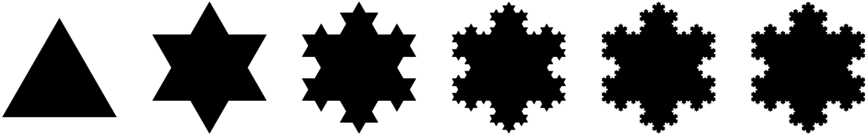
Fractal apertures in laser optics
e.g. J. Christian

Scattering by fractal screens



Lots of mathematical challenges:

- How to formulate well-posed BVPs?
  (What is the right function space setting? How to impose BCs?)
- How do prefractal solutions converge to fractal solutions?
- How can we accurately compute the scattered field?
- ...



Note: several tools developed here might be used in the (numerical) analysis of different IEs & BVPs involving complicated domains.

BVPs & BIEs: long story short...

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These are equivalent to boundary integral equations (BIEs), which can be written as continuous&coercive variational problems

$$\text{find } \phi \in V \quad \text{s.t.} \quad \mathcal{A}(\phi, \psi) = \mathcal{F}(\psi) \quad \forall \psi \in V$$

($\phi = [\partial_n u]$ Neumann jump on Γ) posed in subspaces of $H^{-1/2}(\Gamma_\infty)$:

$$V = \widetilde{H}^{-1/2}(\Gamma) := \overline{C_0^\infty(\Gamma)}^{H^{-1/2}(\mathbb{R}^{n-1})} \qquad \Gamma \text{ open,}$$

$$V = H_\Gamma^{-1/2} := \{u \in H^{-1/2}(\mathbb{R}^{n-1}) : \operatorname{supp} u \subset \Gamma\} \qquad \Gamma \text{ compact.}$$

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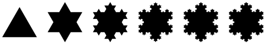
($\widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}}$ if Γ is C^0, or thick..., many cases but $\exists$counterexamples)

How to approximate $\phi \in \begin{matrix} \widetilde{H}^{-1/2}(\Gamma) \\ H_\Gamma^{-1/2} \end{matrix}$ numerically if Γ is rough/fractal?

E.g. Γ hard to mesh, interior is empty, prefractals are not nested...?



Mosco convergence

Key tool is Mosco convergence for closed subspaces of Hilbert H:

Mosco convergence (1969): $H \supset V_j \xrightarrow{\mathcal{M}} V \subset H$ if

- $\forall v \in V, j \in \mathbb{N}, \exists v_j \in V_j$ s.t. $v_j \to v$ (strong approximability)
- $\forall (j_m)$ subseq. of $\mathbb{N}$, $v_{j_m} \in V_{j_m}$, $v_{j_m} \rightharpoonup v$, then $v \in V$ (weak closure)

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Theorem

If $H \supset V_j \xrightarrow{\mathcal{M}} V \subset H$ and sesquilinear form $\mathcal{A}$ is continuous&coercive on H, $\mathcal{F} \in H^*$, then the sequence ϕ_j of solutions of

$$\textit{find } \phi_j \in V_j \quad \textit{s.t.} \quad \mathcal{A}(\phi_j, \psi_j) = \mathcal{F}(\psi_j) \quad \forall \psi_j \in V_j$$

converges (in the norm of H) to the solution of

$$\textit{find } \phi \in V \quad \textit{s.t.} \quad \mathcal{A}(\phi, \psi) = \mathcal{F}(\psi) \quad \forall \psi \in V.$$

We extend this to compactly-perturbed problems.

Mosco convergence in action

If $V_j = \begin{cases} \widetilde{H}^{-1/2}(\Gamma_j) & \Gamma_j \text{ open} \\ H^{-1/2}_{\Gamma_j} & \Gamma_j \text{ comp.} \end{cases}$ $V = \begin{cases} \widetilde{H}^{-1/2}(\Gamma) & \Gamma \text{ open} \\ H^{-1/2}_{\Gamma} & \Gamma \text{ comp.} \end{cases}$ then

$V_j \xrightarrow{\mathcal{M}} V$ implies convergence of prefractal BIE solution to fractal sol:

$\phi_j \to \phi$ in $H^{-1/2}(\Gamma_\infty)$ and $u_j = \mathcal{S}_{\Gamma_*}\phi_j \to u = \mathcal{S}_{\Gamma_*}\phi$ in $W^{1,\mathrm{loc}}(\mathbb{R}^n)$.

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1 open $\Gamma_j \subset \Gamma_{j+1}$ 2 compact $\Gamma_j \supset \Gamma_{j+1}$ 3 non-nested $\Gamma_j \overset{\not\subset}{\not\supset} \Gamma_{j+1}$

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Partition prefractal Γ_j with mesh $M_j = \{T_{j,1}, \ldots, T_{j,N_j}\}$, h_j :=mesh size.
Denote by $V_j^h \subset H^{-1/2}(\Gamma_\infty)$ the space of piecewise constants on M_j.

Then $V_j^h \xrightarrow{\mathcal{M}} V$ implies convergence of Galerkin-BEM solution to ϕ.

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How to choose M_j to ensure convergence?

$$\phi_j^h \xrightarrow{h_j \to 0} \phi_j \xrightarrow{j \to \infty} \phi, \qquad \phi_j^h \xrightarrow{?} \phi$$

Main requirement for Mosco convergence:

strong approximability: $\forall v \in V \; \exists v_j^h \in V_j^h$ s.t. $v_j^h \xrightarrow{H^{-1/2}(\mathbb{R}^{n-1})} v$.

BEM convergence: open screen

Approximation lemma for "pre-convex" meshes

Let $\Pi : L^2(\Omega) \to V^h$ be the orthogonal proj. on pw-constants. Then

$$\|u - \Pi u\|_{\widetilde{H}^s(\Omega)} \le (h/\pi)^{t-s} \|u\|_{H^t(\Omega)}, \qquad \forall u \in H^t(\Omega), \quad -1 \le s \le 0 \le t \le 1.$$

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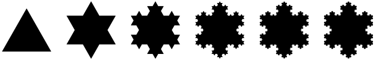
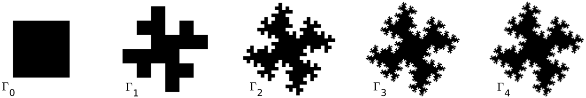
Since $C_0^\infty(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma)$ is dense, this gives convergence for the case of open screen & nested prefractals:



Theorem

Let Γ, Γ_j be bounded open, $\Gamma_j \subset \Gamma_{j+1}$, $\Gamma = \bigcup_{j=0}^\infty \Gamma_j$.
Then BEM convergence holds if $h_j \to 0$ as $j \to \infty$.

Also holds for some non-nested ("sandwiched") $\Gamma_j \underset{\not\supset}{\not\subset} \Gamma_{j+1}$, e.g.



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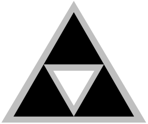
To obtain a smooth approximation we mollify: this enlarges the support.
Currently only results for "thickened prefractals".



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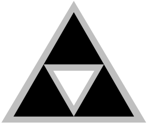

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Let Γ compact & Γ_j open satisfy $\Gamma \subset \Gamma(\epsilon_j) \subset \Gamma_j \subset \Gamma(\eta_j)$, $0 < \epsilon_j < \eta_j \to 0$.
Then BEM convergence holds if $h_j = o(\epsilon_j)$ as $j \to \infty$.
If H_Γ^t is dense in $H_\Gamma^{-1/2}$ for $t \in (-1/2, 0)$ then $h_j = o(\epsilon_j^{-2t})$ suffices.

If Γ is d-set (e.g. IFS attractor), $h_j = o(\epsilon_j^\mu)$, $\mu > n - 1 - \dim_{\rm H}\Gamma$ is enough.

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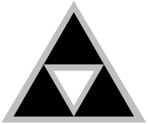

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If Γ is d-set (e.g. IFS attractor), $h_j = o(\epsilon_j^\mu)$, $\mu > n - 1 - \dim_{\mathrm{H}}\Gamma$ is enough.
Proof of (i) (strong approx.): Let $v \in H_\Gamma^t$ and set $v_j := (\psi_{\varepsilon_j/2} * v)$, then

$$\|\Pi_{L^2, V_j^h} v_j - v_j\|_{\widetilde{H}^{-1/2}(\Gamma)} \le (h_j/\pi)^{1/2} \|v_j\|_{L^2(\Gamma_j)} \le (h_j/\pi)^{1/2} (\varepsilon_j/2)^t \|v\|_{H_\Gamma^t}.$$

Open problem: orders of convergence

We cannot prove orders of convergence, yet.
Three obstacles / open questions:

- What is the H^s regularity of the BIE solution $\phi \in \widetilde{H}^{-1/2}(\Gamma)/H_\Gamma^{-1/2}$?

  Conjecture: for Γ a d-set with Hausdorff dimension
  $n-2 < d < n-1$, $u \in H_\Gamma^t$ for $t < (d-n+1)/2 \quad \in (-1/2, 0)$.

- How to ensure quasi-optimality for Mosco convergence?

  (Trivial only for open–nested case )

- How to extend approximation lemma to

$$\|u - \Pi u\|_{\widetilde{H}^s(\Omega)} \leq (h/\pi)^{t-s} \|u\|_{H^t(\Omega)}, \ \forall u \in H^t(\Omega), \quad -1/2 \leq s < t < 0?$$

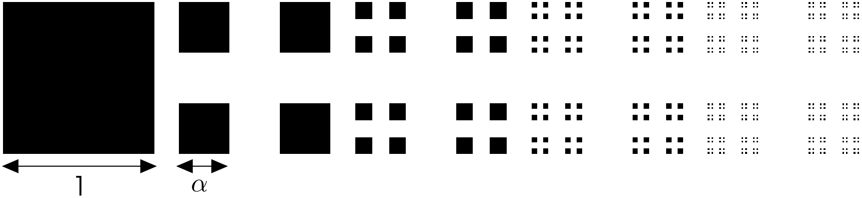
Any suggestion is welcome!

Part II

Examples and numerics

Cantor dust

Cantor dust is Cartesian product of 2 copies of Cantor set with parameter $0 < \alpha < 1/2$. Prefractals $\Gamma_0, \ldots, \Gamma_4$:



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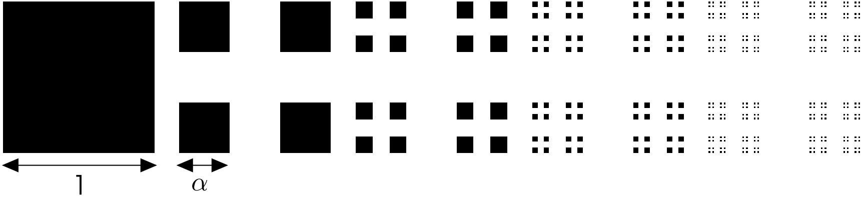
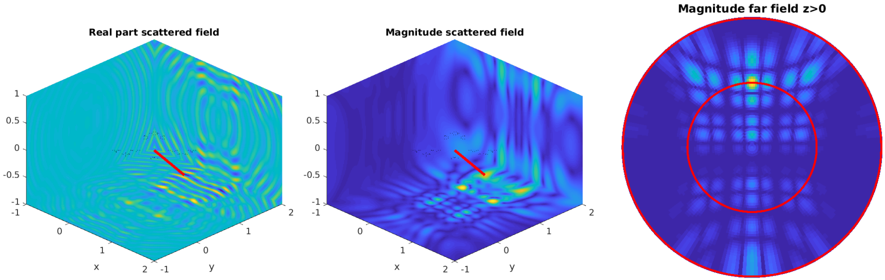


- Γ "audible" ($\phi \neq 0$) $\iff \alpha > \frac{1}{4} \iff \dim_{\mathrm{H}}(\Gamma) > 1$.
  ($\phi \neq 0 \iff \dim_{\mathrm{H}}(\Gamma) > 1$ holds for all d-sets!)
- $H^{-1/2}_{\Gamma_j} \xrightarrow{\mathcal{M}} H^{-1/2}_{\Gamma}$, prefractal solutions ϕ_j converge to ϕ.
- BEM on thickened prefractals converge,
  1 DOF / prefractal component is enough.

Actually BEM converges with even less than 1 DOF/component:
m_j components/element on Γ_j for $1 \leq m_j < 4^{(\frac{\log 4}{\log 1/\alpha} - 1)j}$.

Cantor dust: field plots

Prefractal level $j = 6$, $N_j = 4^6 = 4\,096$ DOFs, $k = 50$, $\alpha = 1/3$.



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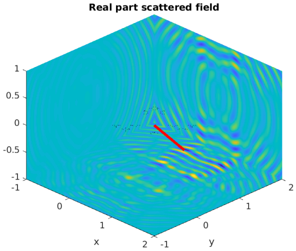
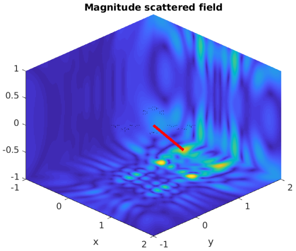
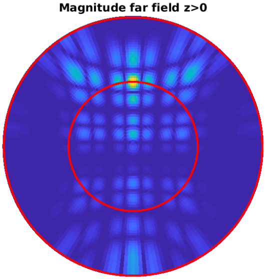
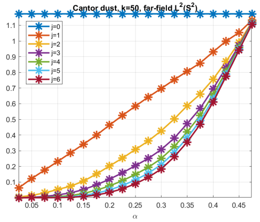


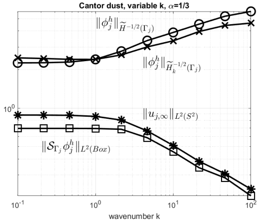






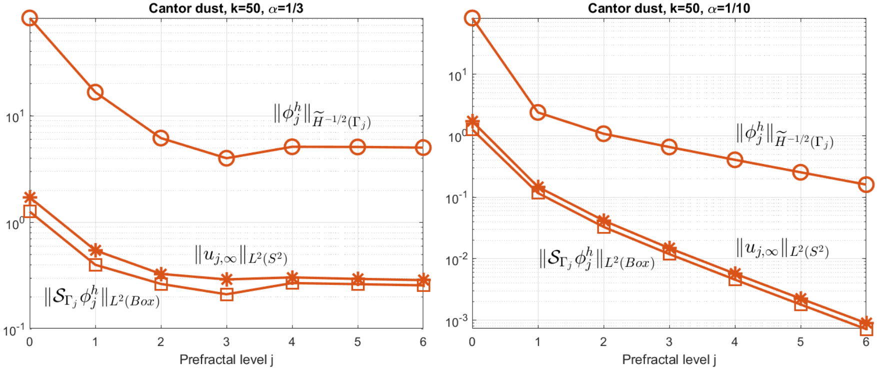
◀ L^2 norms of far-field, $\alpha \in (0.025, 0.475)$, prefractal levels $j = 0, \ldots, 6$.

Solution norms for $\alpha = \frac{1}{3}$ ▶ wavenumber $k \in [0.1, 100]$.



Cantor dust, solution norms

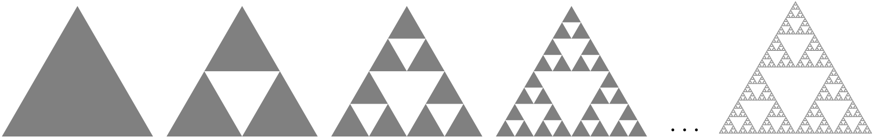
Norm of ◯ Neumann jumps (BIE solution), □ near-field, $*$ far-field:



Norms of the solution on the prefractals converge:

- to positive constant values for $\alpha = 1/3$ (left),
- to 0 for $\alpha = 1/10$ (right).

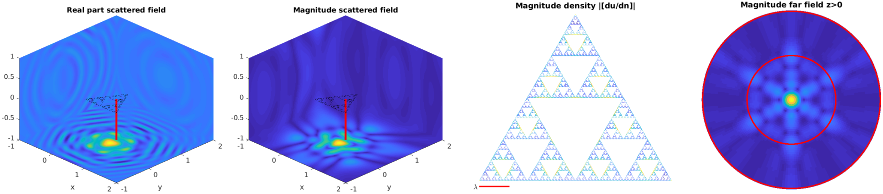
Sierpinski triangle



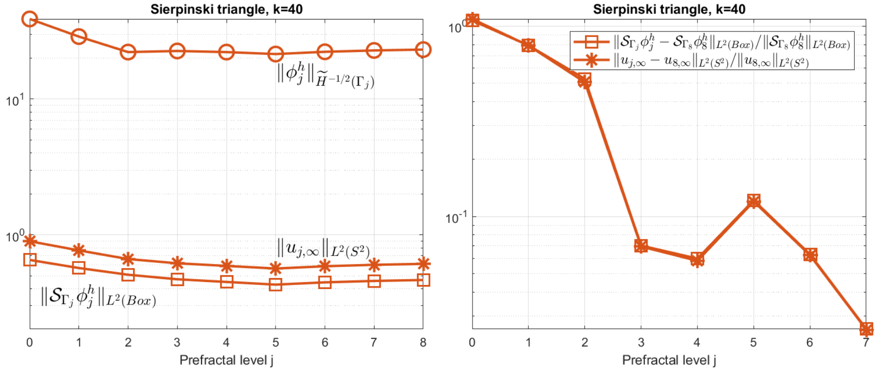
$H^{-1/2}_{\Gamma_j} \xrightarrow{\mathcal{M}} H^{-1/2}_{\Gamma}$, prefractal solutions ϕ_j converge to ϕ.

BEM on thickened prefractals converges if $h_j = o((\frac{3}{4} - \epsilon)^j)$.

Prefractal level $j = 8$, $N_j = 3^8 = 6\,561$ DOFs, $k = 40$:



Sierpinski triangle, solution norms



Right plot near- & far-field:

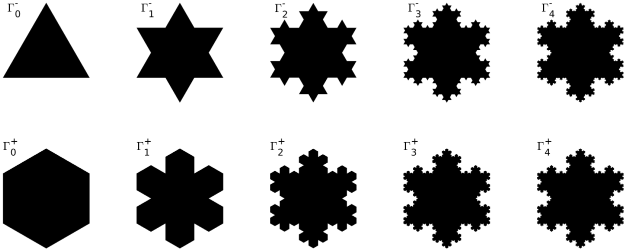
$$\square = \frac{\|\mathcal{S}_{\Gamma_j}\phi_j - \mathcal{S}_{\Gamma_8}\phi_8\|_{L^2(BOX)}}{\|\mathcal{S}_{\Gamma_8}\phi_8\|_{L^2(BOX)}}, \qquad * = \frac{\|u_{j,\infty} - u_{8,\infty}\|_{L^2(\mathbb{S}^2)}}{\|u_{8,\infty}\|_{L^2(\mathbb{S}^2)}}.$$

Prefractal level 3 is where density maxima are located and all wavelength-size prefractal features are resolved: big error reduction!

Koch snowflake

We can approximate Γ from inside and outside with polygons $\Gamma_j^{\pm}$:

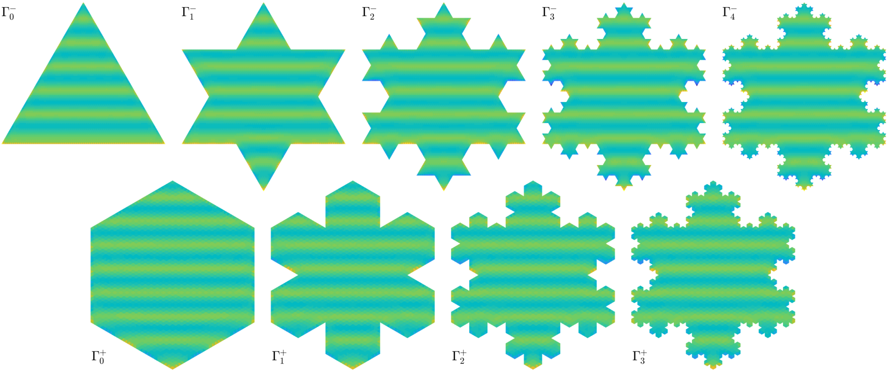
$$\underbrace{\Gamma_1^- \subset \Gamma_2^- \subset \Gamma_3^-}_{\text{open}} \subset \cdots \subset \bigcup_{j\in\mathbb{N}} \Gamma_j^- = \Gamma \subset \overline{\Gamma} = \bigcap_{j\in\mathbb{N}} \Gamma_j^+ \subset \cdots \subset \underbrace{\Gamma_3^+ \subset \Gamma_2^+ \subset \Gamma_1^+}_{\text{closed}}.$$



For a scattering BVP, since Γ is "thick", $\widetilde{H}^{\pm 1/2}(\Gamma) = H^{\pm 1/2}_{\overline{\Gamma}}$
and both sequences $u_j^{\pm}$ converge to the same limit.

(CAETANO + H + M, 2018)

Real part of fields on inner and outer prefractals

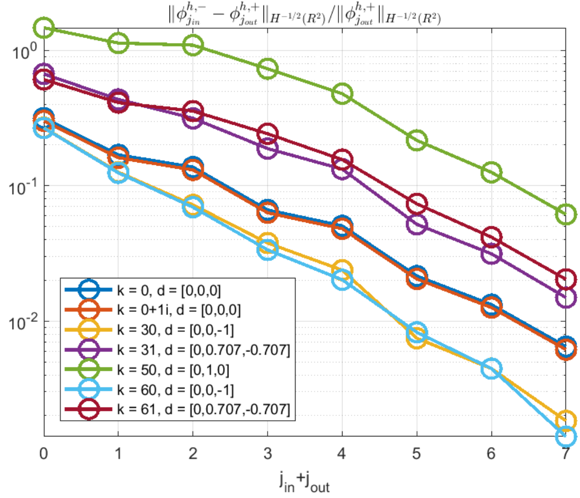


$k = 61$, $\mathbf{d} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^\top$, 3576 to 10344 DOFs.

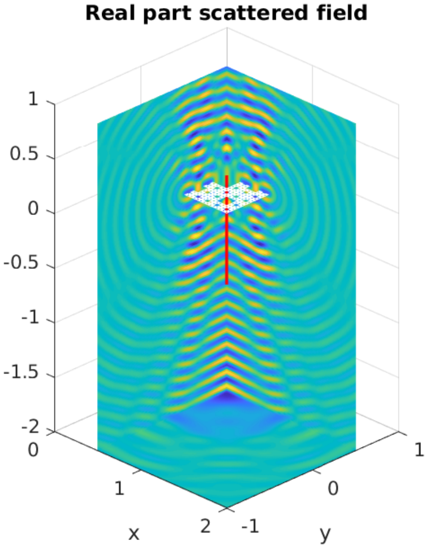
Now I compare $\phi_j^{h,-}$ against $\phi_{j-1}^{h,+}$ and $\phi_j^{h,+}$.

Inner and outer snowflake approximations

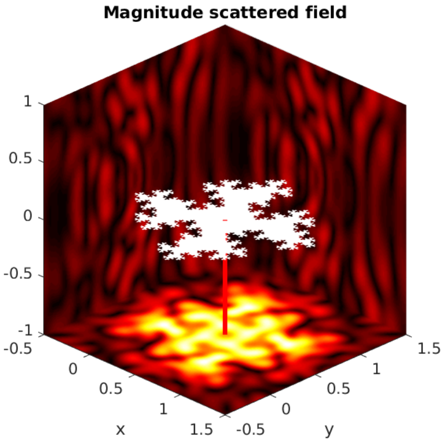
$$\frac{\|\phi^{h,-}_{j_{in}} - \phi^{h,+}_{j_{out}}\|_{H^{-1/2}(\mathbb{R}^2)}}{\|\phi^{h,+}_{j_{out}}\|_{H^{-1/2}(\mathbb{R}^2)}}$$



Other shapes



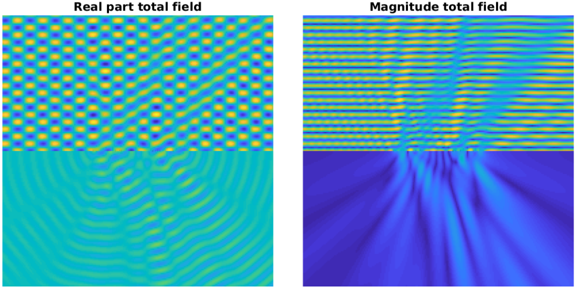
◁ Sierpinski carpet.

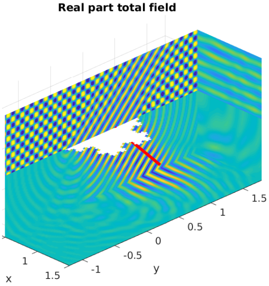


△ "Square snowflake", limit of non-monotonic prefractals.

Apertures

Field through bounded apertures in unbounded Neumann screens computed via Babinet's principle.





$n = 1$, Cantor set $\alpha = 1/3$, prefractal level 12:
field through 0-measure holes!

Koch snowflake-shaped aperture △

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- DPH, AM, *A note on properties of the restriction operator on Sobolev spaces,* JAA 2017.
- SNCW, DPH, *Well-posed PDE and integral equation formulations for scattering by fractal screens,* SIAM J. Math. Anal., 2018.
- A. Caetano, DPH, AM, *Density results for Sobolev, Besov and Triebel-Lizorkin spaces on rough sets* arXiv 2019.
- SNCW, DPH, AM, J. Besson *Boundary element methods for acoustic scattering by fractal screens* coming soon!

. . .

Open questions

- Regularity theory for the fractal solution
- Rates of convergence
- Approximation on fractals
- Fast BEM
- What about curved screens?
  More general rough scatterers?
- What about the Maxwell case?
  Other PDEs? (Laplace, reaction–diffusion already covered.)
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Thank you!

Sobolev spaces on rough subsets of $\mathbb{R}^{n-1}$

We need fractional (Bessel) Sobolev spaces on $\Gamma \subset \mathbb{R}^{n-1}$. For $s \in \mathbb{R}$ let

$$H^s(\mathbb{R}^{n-1}) = \left\{ u \in \mathcal{S}^*(\mathbb{R}^{n-1}) : \|u\|^2_{H^s(\mathbb{R}^{n-1})} := \int_{\mathbb{R}^{n-1}} (1+|\boldsymbol{\xi}|^2)^s |\hat{u}(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} < \infty \right\}$$

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For $\Gamma \subset \mathbb{R}^{n-1}$ open and $F \subset \mathbb{R}^{n-1}$ closed define (McLean)

$$H^s(\Gamma) := \{u|_\Gamma : u \in H^s(\mathbb{R}^{n-1})\} \qquad \text{restriction}$$

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When Γ is Lipschitz it holds that

- $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$ with equal norms
- $s \in \mathbb{N} \Rightarrow \|u\|^2_{H^s(\Gamma)} \sim \sum_{|\alpha| \le s} \int_\Gamma |\partial^\alpha u|^2$
- $\widetilde{H}^s(\Gamma) = H^s_{\Gamma}$ $\qquad (\cong H^s_{00}(\Gamma),\ s \ge 0)$
- $H^{\pm 1/2}_{\partial\Gamma} = \{0\}$
- $\{H^s(\Gamma)\}_{s\in\mathbb{R}}$ and $\{\widetilde{H}^s(\Gamma)\}_{s\in\mathbb{R}}$ are interpolation scales.

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When Γ is Lipschitz it holds that	For general open Γ	
▶ $\widetilde{H}^s(\Gamma) = (H^{-s}(\Gamma))^*$ with equal norms	▶ ✓	
▶ $s \in \mathbb{N} \Rightarrow \|u\|^2_{H^s(\Gamma)} \sim \sum_{\|\alpha\| \le s} \int_\Gamma \|\partial^\alpha u\|^2$	▶ ×	LIPSCHITZ
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BVPs for open and compact screens

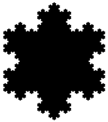
BVP $\mathrm{D}^{op}(\Gamma)$ for open screens

Let $\Gamma \subset \Gamma_\infty$ be bounded & open.
Given $g \in H^{1/2}(\Gamma)$
(for instance, $g = -(\gamma^\pm u^i)|_\Gamma$),
find $u \in C^2(D) \cap W^{1,\mathrm{loc}}(D)$
satisfying

$$\Delta u + k^2 u = 0 \quad \text{in } D,$$

$$(\gamma^\pm u)|_\Gamma = g,$$

Sommerfeld RC.



$\gamma^\pm =$ traces : $W^1(\mathbb{R}^n_\pm) \to H^{1/2}(\Gamma_\infty)$

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Given $g \in \widetilde{H}^{1/2}(\Gamma^c)^\perp$
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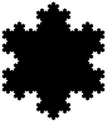

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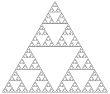

Orthogonal projection
$P_\Gamma : H^{1/2}(\Gamma_\infty) \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$.

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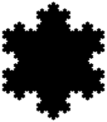

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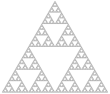

Orthogonal projection
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If Ω bdd open, $\widetilde{H}^{-1/2}(\Omega) = H^{-1/2}_{\overline{\Omega}}$, then D$^{op}(\Omega)$&D$^{co}(\overline{\Omega})$ are equivalent.

Well-posedness & boundary integral equations

Theorem (CW, H, M 2019)

If $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ then problem D$^{op}(\Gamma)$ has a unique solution u.

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u satisfies the representation formula $u(\mathbf{x}) = -\mathcal{S}_\Gamma\phi(\mathbf{x}), \mathbf{x} \in D$,
where $\phi = [\partial_\mathbf{n} u] := \partial_\mathbf{n}^+ u - \partial_\mathbf{n}^- u$ is the unique solution of BIE $S_\Gamma\phi = -g$.

$\mathcal{S}_\Gamma$ = single-layer potential,
S_Γ = single layer operator: cont. & coercive in $H^{-1/2}(\mathbb{R}^{n-1})$ norm.

$\mathcal{S}_\Gamma\psi(\mathbf{x}) := \int_\Gamma \Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{x})\mathrm{d}s(\mathbf{y})$

$\mathcal{S}_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to C^2(D) \cap W^{1,loc}(\mathbb{R}^n)$

$S_\Gamma\psi = (\gamma^\pm \mathcal{S}_\Gamma\psi)|_\Gamma$

$S_\Gamma : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$

$\mathcal{S}_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to C^2(D) \cap W^{1,loc}(\mathbb{R}^n)$

$S_\Gamma = P_\Gamma \gamma^\pm \mathcal{S}_\Gamma$

$S_\Gamma : H_{\overline{\Gamma}}^{-1/2} \to \widetilde{H}^{1/2}(\Gamma^c)^\perp$

Φ is the Helmholtz fundamental solution ($\Phi(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$ for $n = 3$)

When is $\widetilde{H}^{-1/2}(\Gamma) = H_{\overline{\Gamma}}^{-1/2}$?

The previous theorems extend classical results for Lipschitz domains (STEPHAN & WENDLAND 1984, STEPHAN 1987).

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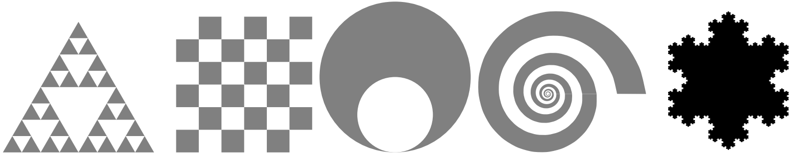
Sufficient conditions for $\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}}$ are that $|\partial\Gamma| = 0$ and either

- Γ is C^0 (e.g. Lipschitz);
- Γ is C^0 except at a set of countably many points $P \subset \partial\Gamma$ such that P has only finitely many limit points;
- Γ is "thick", in the sense of Triebel.



$$(\widetilde{H}^{-1/2}(\Gamma) = H^{-1/2}_{\overline{\Gamma}} \iff C_0^\infty(\Gamma) \overset{\text{dense}}{\subset} \{v \in H^{-1/2}(\mathbb{R}^{n-1}) : \operatorname{supp} v \subset \overline{\Gamma}\})$$

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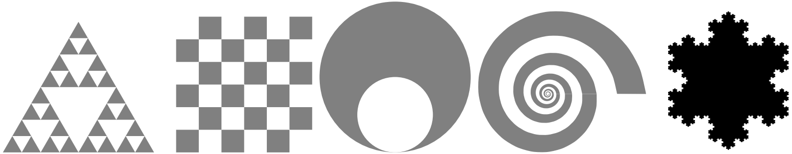

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Cases with $\widetilde{H}^{-1/2}(\Gamma) \neq H^{-1/2}_{\overline{\Gamma}}$ constructed using characterisation:

If $s \in \mathbb{R}$, $\operatorname{int}(\overline{\Gamma})$ is C^0 then $\quad \widetilde{H}^s(\Gamma) = H^s_{\overline{\Gamma}} \iff H^{-s}_{\operatorname{int}(\overline{\Gamma})\setminus\Gamma} = \{0\}$.

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