WAVES 2022, ENSTA PARIS, 25–29 JULY 22

A space–time quasi-Trefftz DG method for the wave equation with smooth coefficients

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arXiv:2011.04617

Part I

Quasi-Trefftz spaces for linear PDEs

Trefftz methods

Consider a boundary value problem

$$\mathcal{L}u = 0 \quad \text{in } D \subset \mathbb{R}^d, \qquad\qquad \mathcal{B}u = g \quad \text{on } \partial D.$$

A Trefftz scheme is a discretisation whose trial (& test) functions v_h are solutions of the PDE $\mathcal{L}v_h = 0$ in each element of a mesh.

This works well for PDEs that

- are linear
- are homogeneous
  (source term is 0)
- have constant coefficients

} $\{v : \mathcal{L}v = 0\}$ is linear space

Trefftz functions are "easy" to build

Examples: $(\mathbf{d} \in \mathbb{R}^n, |\mathbf{d}| = 1)$

Laplace equation $\Delta u = 0 \quad \to \quad$ harmonic polynomials,

Helmholtz equation $\Delta u + k^2 u = 0 \quad \to \quad$ plane waves $e^{ik\mathbf{d}\cdot\mathbf{x}}$,

wave equation $-\Delta u + c^{-2}\partial_t^2 u = 0 \quad \to \quad$ plane waves $f(\mathbf{d}\cdot\mathbf{x} - ct)$.

Quasi-Trefftz methods

What happens if the PDE has smooth coefficients?
We typically don´t know how to construct discrete Trefftz space.

Quasi-Trefftz idea:
use discrete functions that are approximate solution of the PDE
$\mathcal{L}v_h \approx 0$ (in each mesh element K).

More precisely: quasi-Trefftz functions v_h satisfy

$$(D^{\boldsymbol{i}}\mathcal{L}v_h)(\mathbf{x}_K) = 0 \quad \forall \boldsymbol{i} \in \mathbb{N}_0^n,\ |\boldsymbol{i}| \le q, \qquad \text{for a given } \mathbf{x}_K \in K,\ q \in \mathbb{N}.$$

Instead of $\mathcal{L}v_h = 0$ in K, this only requires that the
degree-q Taylor polynomial (centred at a given point $\mathbf{x}_K$) of $\mathcal{L}v_h$ is 0.

$$\Rightarrow \text{ Small residual:} \qquad \mathcal{L}v_h(\mathbf{x}) = \mathcal{O}(|\mathbf{x} - \mathbf{x}_K|^{q+1}), \qquad \mathbf{x} \in K.$$

Which kind of functions are these?

Polynomial quasi-Trefftz approximation

Let m be the order of the linear PDE operator $\mathcal{L}$.
We use degree-p polynomials: for $p \in \mathbb{N}$

$$\mathbb{QT}^p_{\mathcal{L}}(K) := \Big\{ v_h \in \mathbb{P}^p(K) : \ (D^{\boldsymbol{i}} \mathcal{L} v_h)(\mathbf{x}_K) = 0 \ \ \forall \boldsymbol{i} \in \mathbb{N}_0^n, \ |\boldsymbol{i}| \le p - m \Big\}.$$

Taylor polynomials of PDE solutions are quasi-Trefftz

Let $\mathcal{L} = \sum_{|\boldsymbol{j}| \le m} \alpha_{\boldsymbol{j}} D^{\boldsymbol{j}}$ for $\alpha_{\boldsymbol{j}} \in C^{\max\{p-m,0\}}(K)$, $\quad \mathcal{L}u = 0$ for $u \in C^{p+1}(K)$.
Then $T^{p+1}_{\mathbf{x}_K}[u] \in \mathbb{QT}^p_{\mathcal{L}}(K)$. (Degree-$p$ Taylor p.)

h-approximation estimates follow for any (linear, smooth-coeff.) PDE:

$\mathcal{L}$ and u as above, $\quad K$ star-shaped wrt $\mathbf{x}_K$,
$r_K := \sup_{\mathbf{x} \in K} |\mathbf{x} - \mathbf{x}_K|$

$$\inf_{P \in \mathbb{QT}^p_{\mathcal{L}}(K)} |u - P|_{C^q(K)} \le \frac{d^{p+1-q}}{(p+1-q)!} r_K^{p+1-q} |u|_{C^{p+1}(K)} \qquad \forall q \le p.$$

Quasi-Trefftz methods

$\mathbb{QT}^p_\mathcal{L}$ has same approximation orders as full polynomial space $\mathbb{P}^p$ but much fewer DOFs. Typically, on $K \subset \mathbb{R}^d$:

$$\dim(\mathbb{QT}^p_\mathcal{L}) = \mathcal{O}_{p\to\infty}(p^{d-1}) \quad \ll \quad \dim(\mathbb{P}^p) = \mathcal{O}_{p\to\infty}(p^d).$$

To approximate a BVP we also need:

- a (DG) variational formulation,
- a basis of $\mathbb{QT}^p_\mathcal{L}$.

} In the rest of the talk, for the wave eq. only.

Missing:

- p-estimates,
- estimates in Sobolev norms.

Part II

Space–time DG for the wave equation

Initial–boundary value problem

Wave eq.: $-\Delta u + c^{-2}\partial_t^2 u = 0$. Set $v = \partial_t u$ and $\boldsymbol{\sigma} = -\nabla u$.

First-order initial–boundary value problem (Dirichlet): find $(v, \boldsymbol{\sigma})$ s.t.

$$\begin{cases} \nabla v + \partial_t \boldsymbol{\sigma} = \mathbf{0} & \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1},\ n \in \mathbb{N}, \\ \nabla \cdot \boldsymbol{\sigma} + \dfrac{1}{c^2}\partial_t v = 0 & \text{in } Q, \\ v(\cdot,0) = v_0, \quad \boldsymbol{\sigma}(\cdot,0) = \boldsymbol{\sigma}_0 & \text{on } \Omega, \\ v(\mathbf{x},\cdot) = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

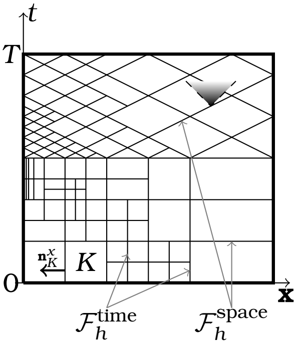
Velocity $c = c(\mathbf{x})$ piecewise smooth. $\Omega \subset \mathbb{R}^n$ Lipschitz bounded.

Extensions:

- Neumann $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ & Robin $\frac{\vartheta}{c} v - \boldsymbol{\sigma} \cdot \mathbf{n} = g$ BCs
- more general coeff.'s $-\nabla \cdot (\rho^{-1}\nabla u) + G\,\partial_t^2 u = 0$
- ▷ Maxwell equations
- ▷ elasticity
- ▷ 1^{st} order hyperbolic systems. . .

Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on Q.
Assume: $c = c(\mathbf{x})$ smooth in each element.



Assume: each face $F = \partial K_1 \cap \partial K_2$
with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c|\mathbf{n}_F^x| < n_F^t$ $\qquad F \subset \mathcal{F}_h^{\text{space}}$

  or
- time-like: $n_F^t = 0$ $\qquad F \subset \mathcal{F}_h^{\text{time}}$

Usual DG notation with averages $\{\!\{\cdot\}\!\}$,
$\mathbf{n}^x$-normal space jumps $[\![\cdot]\!]_{\mathbf{N}}$, $\quad n^t$-time jumps $[\![\cdot]\!]_t$.
Lateral boundary $\mathcal{F}_h^{\partial} := \partial\Omega \times (0, T)$.

DG elemental equation and numerical fluxes

Multiply PDEs with test field $(w, \boldsymbol{\tau})$ & integrate by parts on $K \in \mathcal{T}_h$:

$$- \int_K \left(v\Big(\nabla \cdot \boldsymbol{\tau} + c^{-2}\partial_t w\Big) + \boldsymbol{\sigma} \cdot \Big(\nabla w + \partial_t \boldsymbol{\tau}\Big)\right) \mathrm{d}V$$
$$+ \int_{\partial K} \left((v\,\boldsymbol{\tau} + \boldsymbol{\sigma}\, w) \cdot \mathbf{n}_K^x + \Big(\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + c^{-2} v\, w\Big)\, n_K^t \right) \mathrm{d}S = 0.$$

This is an "ultra-weak" variational formulation (UWVF).

Approximate skeleton traces of $(v, \boldsymbol{\sigma})$ with numerical fluxes $(\widehat{v}_h, \widehat{\boldsymbol{\sigma}}_h)$, defined as $\qquad \alpha, \beta \in L^\infty(\mathcal{F}_h^{\mathrm{time}} \cup \mathcal{F}_h^{\partial})$

$$\widehat{v}_h := \begin{cases} v_h^- \\ v_0 \\ \{\!\{v_h\}\!\} + \beta[\![\boldsymbol{\sigma}_h]\!]_{\mathbf{N}} \\ g \end{cases} \quad \widehat{\boldsymbol{\sigma}}_h := \begin{cases} \boldsymbol{\sigma}_h^- & \text{on } \mathcal{F}_h^{\mathrm{space}} \cup \mathcal{F}_h^T \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^0 \\ \{\!\{\boldsymbol{\sigma}_h\}\!\} + \alpha[\![v_h]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\mathrm{time}} \\ \boldsymbol{\sigma}_h - \alpha(v - g)\mathbf{n}_\Omega^x & \text{on } \mathcal{F}_h^{\partial} \end{cases}$$

"upwind in time, elliptic-DG in space".
$\alpha = \beta = 0 \to$ KRETZSCHMAR–S.–T.–W., $\quad \alpha\beta \geq \frac{1}{4} \to$ MONK–RICHTER.

Space–time DG formulation

Substitute the fluxes in the elemental equation,
add volume penalty term as in (IMBERT-GÈRARD, MONK 2017),
choose discrete space $\mathbf{V}_p \subset H^1(\mathcal{T}_h)^{1+n}$, sum over K $\rightarrow$ write **x**t-DG:

Seek $(v_h, \boldsymbol{\sigma}_h) \in \mathbf{V}_p$ s.t., $\forall (w, \boldsymbol{\tau}) \in \mathbf{V}_p$,
$\mathcal{A}(v_h, \boldsymbol{\sigma}_h; w, \boldsymbol{\tau}) = \ell(w, \boldsymbol{\tau})$ where...

$$
\begin{aligned}
\mathcal{A}(v_h, \boldsymbol{\sigma}_h; w, \boldsymbol{\tau}) :=& - \sum_{K \in \mathcal{T}_h} \int_K \Big(v_h \big(\nabla \cdot \boldsymbol{\tau} + c^{-2} \partial_t w \big) + \boldsymbol{\sigma}_h \cdot \big(\nabla w + \partial_t \boldsymbol{\tau} \big) \Big) \, \mathrm{d}V \\
&+ \int_{\mathcal{F}_h^{\text{space}}} \Big(\frac{v_h^- [\![w]\!]_t}{c^2} + \boldsymbol{\sigma}_h^- \cdot [\![\boldsymbol{\tau}]\!]_t + v_h^- [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \boldsymbol{\sigma}_h^- \cdot [\![w]\!]_{\mathbf{N}} \Big) \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{\text{time}}} \Big(\{\!\{ v_h \}\!\} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \{\!\{ \boldsymbol{\sigma}_h \}\!\} \cdot [\![w]\!]_{\mathbf{N}} + \alpha [\![v_h]\!]_{\mathbf{N}} \cdot [\![w]\!]_{\mathbf{N}} + \beta [\![\boldsymbol{\sigma}_h]\!]_{\mathbf{N}} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \Big) \, \mathrm{d}S \\
&+ \int_{\Omega \times \{T\}} (c^{-2} v_h w + \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}) \, \mathrm{d}S \qquad + \int_{\mathcal{F}_h^{\partial}} (\boldsymbol{\sigma}_h \cdot \mathbf{n}_\Omega + \alpha v_h) w \, \mathrm{d}S \\
&+ \sum_{K \in \mathcal{T}_h} \int_K \Big(\mu_1 (\nabla \cdot \boldsymbol{\sigma} + c^{-2} \partial_t v)(\nabla \cdot \boldsymbol{\tau} + c^{-2} \partial_t w) + \mu_2 (\partial_t \boldsymbol{\sigma} + \nabla v) \cdot (\partial_t \boldsymbol{\tau} + \nabla w) \Big) \, \mathrm{d}V,
\end{aligned}
$$

$$
\ell(w, \boldsymbol{\tau}) := \int_{\Omega \times \{0\}} (c^{-2} v_0 w + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^{\partial}} g (\alpha w - \boldsymbol{\tau} \cdot \mathbf{n}_\Omega) \, \mathrm{d}S.
$$

Coercivity in DG skeleton norm

Key property, from integration by parts, is coercivity in DG norm:

$$\mathcal{A}(w,\boldsymbol{\tau};w,\boldsymbol{\tau}) \geq |||(w,\boldsymbol{\tau})|||_{\mathrm{DG}}^2 \qquad \forall (w,\boldsymbol{\tau}) \in \prod_{K\in\mathcal{T}_h} H^1(K)^{n+1}$$

$$\begin{aligned}|||(w,\boldsymbol{\tau})|||_{\mathrm{DG}}^2 := {} & \frac{1}{2}\left\|\left(\frac{1-\gamma}{n_F^t}\right)^{1/2} c^{-1}[\![w]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{space}})} + \frac{1}{2}\left\|\left(\frac{1-\gamma}{n_F^t}\right)^{1/2}[\![\boldsymbol{\tau}]\!]_t\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{space}})^n}\\
& + \frac{1}{2}\left\|c^{-1}w\right\|^2_{L^2(\mathcal{F}_h^0\cup\mathcal{F}_h^T)} + \frac{1}{2}\left\|\boldsymbol{\tau}\right\|^2_{L^2(\mathcal{F}_h^0\cup\mathcal{F}_h^T)^n}\\
& + \left\|\alpha^{1/2}[\![w]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})^n} + \left\|\beta^{1/2}[\![\boldsymbol{\tau}]\!]_{\mathbf{N}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})} + \left\|\alpha^{1/2}w\right\|^2_{L^2(\mathcal{F}_h^{\partial})}\\
& + \sum_{K\in\mathcal{T}_h}\left(\left\|\mu_1^{1/2}(c\nabla\cdot\boldsymbol{\tau}+c^{-1}\partial_t w)\right\|^2_{L^2(K)} + \left\|\mu_2^{1/2}(\nabla w+\partial_t\boldsymbol{\tau})\right\|^2_{L^2(K)^n}\right)\end{aligned}$$

$\gamma := \frac{\|c\|_{C^0(F)}|\mathbf{n}_F^x|}{n_F^t} \in [0,1) \sim$ distance between space-like face F & char. cone.

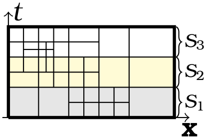
▶ Well-posedness and quasi-optimality (∀ discrete spaces)

$$|||(v,\boldsymbol{\sigma})-(v_h,\boldsymbol{\sigma}_h)|||_{\mathrm{DG}} \leq 3\inf_{(w,\boldsymbol{\tau})\in\mathbf{V}_p}|||(v,\boldsymbol{\sigma})-(w,\boldsymbol{\tau})|||_{\mathrm{DG}^+}$$

Global, implicit and explicit $\mathbf{x}t$-DG schemes

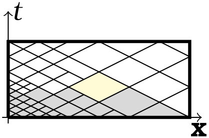
1 $\mathbf{x}t$-DG formulation is global in space–time domain Q:
▶ huge linear system! Good for adaptivity, DD...

2 If mesh is partitioned in time-slabs $\Omega \times (t_{j-1}, t_j)$
then matrix is block lower-triangular:
sequentially solve a system for each slab
▶ implicit method.



3 If mesh is "tent-pitched", DG solution
is computed with a sequence of local systems:
▶ explicit method, allows parallelism!

ÜNGÖR–SHEFFER, MONK–RICHTER...



Versions 1–2–3 are algebraically equivalent (on the same mesh).

Related works on $\mathbf{x}t$-DG formulations

Proposed $\mathbf{x}t$-DG formulation comes from:

- MONK, RICHTER 2005, linear symmetric hyperbolic systems, tent-pitched meshes, $\mathbb{P}^p$ spaces, $\alpha\beta \geq \frac{1}{4}$
- KRETZSCHMAR, SCHNEPP ET AL. 2014–16 Maxwell eq.s, Trefftz
- M., PERUGIA 2018 Trefftz error analysis
- PERUGIA, SCHOEBERL, STOCKER, WINTERSTEIGER 2020 Trefftz & tents

- IMBERT-GÉRARD, M., STOCKER 2020 — arXiv:2011.04617 pw-smooth c, quasi-Trefftz

Related works:

- BANSAL, M., PERUGIA, SCHWAB 2021 corner sing.s, sparse grids
- BARUCQ, CALANDRA, DIAZ, SHISHENINA 2020 Trefftz + elasticity
- GÓMEZ, M. 2021 Trefftz + Schrödinger

Many other $\mathbf{x}t$-DG formulations for waves exist!

Part III

Quasi-Trefftz bases for the wave equation

Quasi-Trefftz space

Define wave operator $\square_G u := \Delta u - G\partial_t^2 u$, $G(\mathbf{x}) = c^{-2}$ smooth.
Fix $(\mathbf{x}_K, t_K) \in K \subset \mathbb{R}^{n+1}$. Quasi-Trefftz (polynomial) space:

$$\mathbb{QU}^p(K) := \left\{u \in \mathbb{P}^p(K) : \quad D^{\boldsymbol{i}}\square_G u(\mathbf{x}_K, t_K) = 0, \quad \forall |\boldsymbol{i}| \le p-2\right\}$$

$$\mathbb{QW}^p(K) := \left\{(\partial_t u, -\nabla u), u \in \mathbb{QU}^{p+1}(K)\right\}$$

- Taylor polynomials of smooth wave solutions belong to $\mathbb{QU}^p(K)$
- $\mathbf{x}t$-DG is quasi-optimal

It follows that $\mathbf{x}t$-DG converges with optimal rates in DG norm:

$$|||(v, \boldsymbol{\sigma}) - (v_h, \boldsymbol{\sigma}_h)|||_{\mathrm{DG}} \le C \sup_{K\in\mathcal{T}_h} h_{K,c}^{p+1/2} \; |u|_{C_c^{p+2}(K)}$$

Quasi-Trefftz basis

The local discrete space is clear.
How to construct a basis for it? Use the following fact:

$u \in \mathbb{QU}^p(K)$ is determined by $u(\cdot, t_K)$ and $\partial_t u(\cdot, t_K)$

Choose two $\mathbf{x}$-only polynomial basis:

$$\{\widehat{b}_J\}_{J=1,\dots,\binom{p+n}{n}} \text{ for } \mathbb{P}^p(\mathbb{R}^n), \qquad \{\widetilde{b}_J\}_{J=1,\dots,\binom{p-1+n}{n}} \text{ for } \mathbb{P}^{p-1}(\mathbb{R}^n).$$

Construct a basis for $\mathbb{QU}^p(K)$ "evolving" $\widehat{b}_J$ and $\widetilde{b}_J$ in time:

$$\left\{ b_J \in \mathbb{QU}^p(K) : \begin{array}{lll} b_J(\cdot, t_K) = \widehat{b}_J, & \partial_t b_J(\cdot, t_K) = 0, & \text{for } J \le \binom{p+n}{n} \\ b_J(\cdot, t_K) = 0, & \partial_t b_J(\cdot, t_K) = \widetilde{b}_{J-\binom{p+n}{n}}, & \text{for } \binom{p+n}{n} < J \end{array} \right\}$$

for $J = 1, \dots, \binom{p+n}{n} + \binom{p-1+n}{n}$.

We prove that this defines a basis and show how to compute $\{b_J\}$.

Computation of basis coefficients

Fix $n = 1$ (for simplicity). Denote $G(x) = \sum_{m=0}^{\infty} g_m (x - x_K)^m$. $g_0 > 0$.
Monomial expansion of basis element:

$$b_J(x,t) = \sum_{i_x + i_t \le p} a_{i_x,i_t} (x - x_K)^{i_x} (t - t_K)^{i_t},$$

Cauchy conditions $(b_J(\cdot, t_K), \partial_t b_J(\cdot, t_K))$ determine $a_{i_x,0}$, $a_{i_x,1}$.

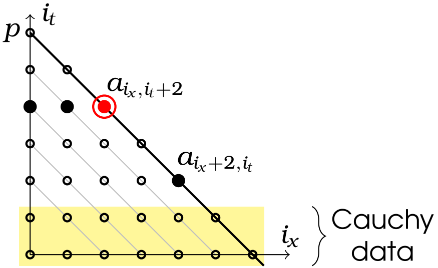
$b_J \in \mathbb{QU}^p$ if coeff.s satisfy: for $i_x + i_t \le p - 2$

$$\partial_x^{i_x} \partial_t^{i_t} \Box_G b_J(x_K, t_K) = (i_x + 2)!\, i_t!\, a_{i_x+2,i_t} - \sum_{j_x=0}^{i_x} i_x!\, (i_t + 2)!\, g_{i_x - j_x}\, a_{j_x,i_t+2} \stackrel{!}{=} 0$$

Linear system for coeff.s a_{i_x,i_t}.

Compute a_{i_x,i_t+2} ◉
from coefficients ● :

first loop across diagonals ↗,
then along diagonals ↖.



Basis construction: algorithm — $n = 1$

Data: $(g_m)_{m\in\mathbb{N}_0}$, x_K, t_K, p.

Choose favourite polynomial bases $\{\widehat{b}_J\}$, $\{\widetilde{b}_J\}$ in x,
$\to$ coeff's $a_{k_x,0}$, $a_{k_x,1}$.

For each J (i.e. for each basis function), construct b_J as follows:

for $\ell = 2$ *to* p *(loop across diagonals ↗)* **do**

  for $i_t = 0$ *to* $\ell - 2$ *(loop along diagonals ↖)* **do**

    set $i_x = \ell - i_t - 2$ and compute

$$a_{i_x,i_t+2} = \frac{(i_x+2)(i_x+1)}{(i_t+2)(i_t+1)g_0} a_{i_x+2,i_t} - \sum_{j_x=0}^{i_x-1} \frac{g_{i_x-j_x}}{g_0} a_{j_x,i_t+2}$$

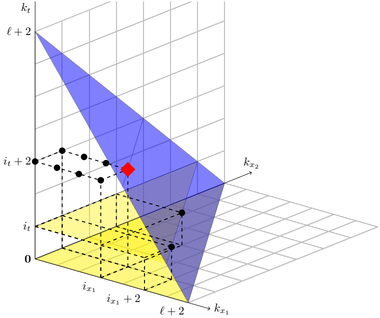
  end

end

$$b_J(x,t) = \sum_{0<k_x+k_t\le p} a_{k_x,k_t}(x-x_K)^{k_x}(t-t_K)^{k_t}$$

Basis construction: algorithm — $n > 1$

In higher space dimensions $n > 1$,
with $G(\mathbf{x}) = \sum_{\boldsymbol{i}_\mathbf{x}} (\mathbf{x} - \mathbf{x}_K)^{\boldsymbol{i}_\mathbf{x}} g_{\boldsymbol{i}_\mathbf{x}}$,
the algorithm is the same
with a further inner loop:



for $\ell = 2$ *to* p *(loop across* $\{|\boldsymbol{i}_\mathbf{x}| + i_t = \ell - 2\}$ *hyperplanes,* $\nearrow$*)* **do**

  for $i_t = 0$ *to* $\ell - 2$ *(loop across constant-t hyperplanes* $\uparrow$*)* **do**

    for $\boldsymbol{i}_\mathbf{x}$ *with* $|\boldsymbol{i}_\mathbf{x}| = \ell - i_t - 2$ **do**

$$a_{\boldsymbol{i}_\mathbf{x}, i_t+2} = \sum_{l=1}^{n} \frac{(i_{x_l}+2)(i_{x_l}+1)}{(i_t+2)(i_t+1)g_\mathbf{0}} a_{\boldsymbol{i}_\mathbf{x}+2\mathbf{e}_l, i_t} - \sum_{\boldsymbol{j}_\mathbf{x} < \boldsymbol{i}_\mathbf{x}} \frac{g_{\boldsymbol{i}_\mathbf{x}-\boldsymbol{j}_\mathbf{x}}}{g_\mathbf{0}} a_{\boldsymbol{j}_\mathbf{x}, i_t+2}$$

    end

  end

end

More general IBVPs

Everything extends to 2 piecewise-smooth material parameters ρ, G:

$$\nabla v + \rho \partial_t \boldsymbol{\sigma} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + G \partial_t v = 0,$$

Wavespeed is $c = (\rho G)^{-1/2}$.

Second-order version:

$$-\nabla \cdot \left(\frac{1}{\rho} \nabla u \right) + G\, \partial_t^2 u = 0 \qquad \left(v = \partial_t u,\ \boldsymbol{\sigma} = -\frac{1}{\rho} \nabla u \right).$$

Basis coefficient algorithm needs some more terms.

If the 1st-order IBVP does not come from a 2nd-order one, we use

$$\mathbb{QT}^p(K) := \left\{ (w, \boldsymbol{\tau}) \in \mathbb{P}^p(K)^{n+1} \;\middle|\; \begin{array}{r} D^{\boldsymbol{i}}(\nabla w + \rho \partial_t \boldsymbol{\tau})(\mathbf{x}_K, t_K) = \mathbf{0} \\ D^{\boldsymbol{i}}(\nabla \cdot \boldsymbol{\tau} + G \partial_t w)(\mathbf{x}_K, t_K) = 0 \\ \forall |\boldsymbol{i}| \le p - 1 \end{array} \right\}$$

This space is only slightly larger ($\approx \frac{n+1}{2} \times$, still $\mathcal{O}_{p\to\infty}(p^n)$ DOFs) and allows the same analysis.

Part IV

Numerical experiments

Numerics

- Implemented in NGSolve.

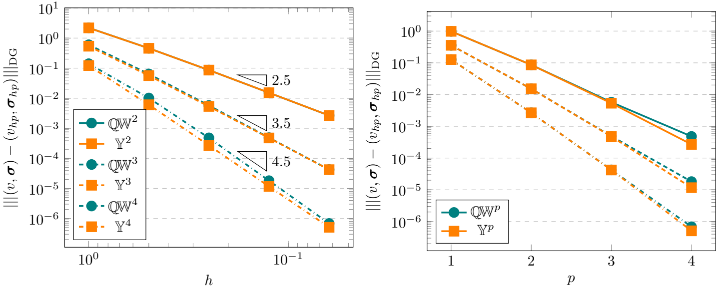
  `https://github.com/PaulSt/NGSTrefftz`
- Both Cartesian and tent-pitched meshes.
- Volume penalty term not needed in computations.
- DG flux coefficients $\alpha^{-1} = \beta = c$, but even $\alpha = \beta = 0$ works.
- Good conditioning.
- Monomial bases $\{\widehat{b}_J\}, \{\widetilde{b}_J\}$ outperform Legendre/Chebyshev.

Numerics 1: convergence

Compare quasi-Trefftz and full polynomials spaces

$$\mathbb{QW}^p(\mathcal{T}_h) := \Pi_K\big\{(\partial_t u, -\nabla u),\ u \in \mathbb{QU}^{p+1}(K)\big\}$$
$$\mathbb{Y}^p(\mathcal{T}_h) := \Pi_K\big\{(\partial_t u, -\nabla u),\ u \in \mathbb{P}^{p+1}(K)\big\}$$

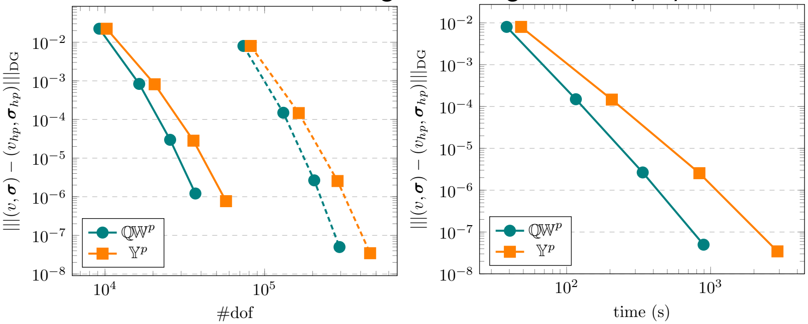
DG-norm error: optimal order in h, exponential in p.



$n = 2, \quad G = (x_1 + x_2 + 1)^{-1}, \quad u = (x_1 + x_2 + 1)^{2.5} e^{-\sqrt{7.5}t}, \quad Q = (0,1)^3.$

Numerics 2: DOFs & computational time

Quasi-Trefftz wins > 1 order of magnitude against full polynomials:

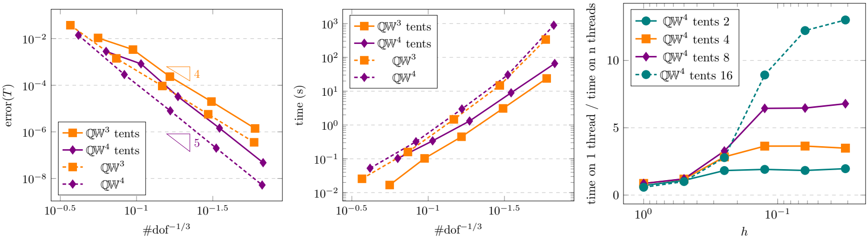


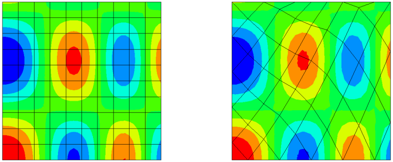
$h = 2^{-3}, 2^{-4}, \quad p = 1, 2, 3, 4.$

$n = 2, \quad G = x_1 + x_2 + 1, \quad u = \mathrm{Ai}(-x_1 - x_2 - 1)\cos(\sqrt{2}t), \quad Q = (0,1)^3.$

Numerics 3: tent pitching

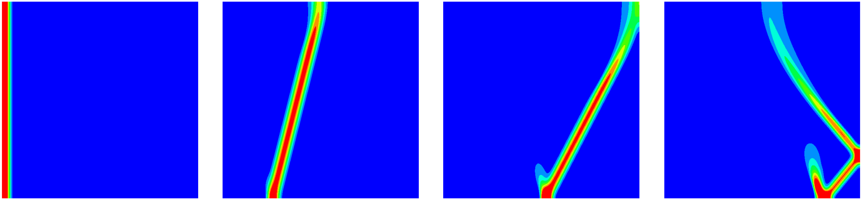
($n = 2$) Final-time error, computational time (sequential), speedup:
(#dof$^{-1/3} \sim h$)



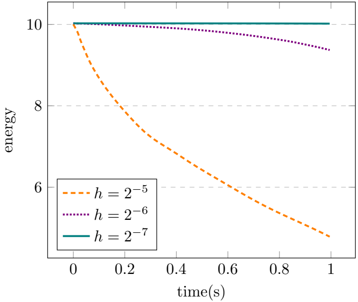


Numerics 4: energy conservation

Plane wave through medium with $G = 1 + x_2$ in $(0, 1)^3$:



$$\mathcal{E} = \frac{1}{2} \int_\Omega (c^{-2} v^2 + |\boldsymbol{\sigma}|^2) \, \mathrm{d}S$$



DG scheme is (provably) dissipative.
For $p = 3$, $h = 2^{-7}$, only 0.076% loss.

Numerics 5: rough solutions

$v_0(x) = \boldsymbol{\sigma}_0(x) = \max(0.25 - |x|, 0) =$  $\in H^1(\Omega) \setminus C^1(\Omega)$,

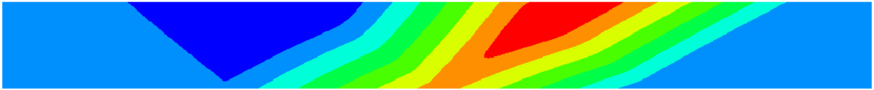
$G(x) = (1+x)^{-2}$, $\quad \rho = 1$, $\quad c = 1 + x$, $\quad$ on $\Omega = (-0.5, 0.5)$.

h	$L^2(\Omega \times \{T\})^2$ error	rates
2^{-6}	0.020	
2^{-7}	0.012	0.73
2^{-8}	0.0068	0.82
2^{-9}	0.0037	0.88
2^{-10}	0.0018	1.0

$\mathbb{QW}^0(\mathcal{T}_h)$ (piecewise-constants) on uniform Cartesian meshes.

Optimal $\mathcal{O}(h)$ convergence even for $\quad u \in H^2(\mathcal{T}_h) \setminus C^2(\mathcal{T}_h)$.

v :



$\boldsymbol{\sigma}$:

Summary

Quasi-Trefftz DG:

- Extend Trefftz scheme to piecewise-smooth coefficients.
  Basis are PDE solution "up to given order in h".
- Simple construction of basis functions:
  same "Cauchy data" at element centre as for Trefftz.
- Use in $\mathbf{x}t$-DG , stability and error analysis.
  High orders of convergence in h,
  much fewer DOFs than standard polynomial spaces.

If you use DG for linear PDEs, try quasi-Trefftz & save DOFs!

IMBERT-GÉRARD, M., STOCKER, arXiv:2011.04617
https://github.com/PaulSt/NGSTrefftz

Thank you!