Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Consider the Cauchy Problem

$$\begin{cases} y' = 2y - y^2 \\ y(0) = y_0 \end{cases}$$

Find the main properties of the solutions and draw their qualitative graph, as y_0 ranges in $[0, +\infty)$. Moreover, determine explicitely the solutions in case $y_0 \in [0, 2]$.

2. Given the ODE system

$$\begin{cases} x' = x + \alpha^2 y \\ y' = \frac{1}{\alpha} x + \alpha y \end{cases}$$

find the values of $\alpha \in \mathbb{R}$ such that:

a) all solutions are bounded in $[0, +\infty)$;

b) there exist solutions (different from the solution identically equal to 0) bounded in $[0, +\infty)$

making explicit the form of the solutions in all the cases.

3. Given the sequence of functions for $n \geq 2$ and $x \in (0, +\infty)$:

$$f_n(x) = \frac{\log x}{\arctan(x^{1/n}) + x^n}$$

a) verify that $f_n \in L^1(0, +\infty)$ for every $n \geq 2$;

b) find f such that $f_n \to f$ pointwise in $(0, +\infty)$;

c) verify that $f_n \to f$ in $L^1(0, +\infty)$ (use the Lebesgue dominated convergence theorem);

d) compute the $\displaystyle\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)\, dx$

4. Using the method of separation of variables, determine the solution of the following Initial-Boundary Value Problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{for } (x,t) \in (0,\ell) \times \mathbb{R}^+ \\ u(x,0) = f(x) & \text{for } x \in (0,\ell) \\ u(0,t) = u(\ell,t) = 0 & \text{for } t \in \mathbb{R}^+ \end{cases}$$

where $\ell \in \mathbb{R}^+$, $f \in C^\infty([0,\ell] \times \mathbb{R})$. Make explicit the compatibility conditions needed on f in order to solve the problem. Prove then the uniqueness of solutions.