Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Let $y_0 \in \mathbf{R}$ and consider the following Cauchy Problem

$$\begin{cases} y'(x) = \dfrac{2x}{x^2-1}(y-1) \\ y(2) = y_0. \end{cases}$$

1.1) Discuss local and global existence and uniqueness of solutions.

1.2) Draw a qualitative graph of solutions.

1.3) Find the explicit solutions (with the respective domains) depending from y_0.

1.4) Find the explicit solutions (with the respective domain) in case $y_0 = 3$.

2. Let $y(x)$ be the solution of the following Cauchy problem

$$\begin{cases} y''(x) + 2y'(x) + y(x) = 8e^x - 5 \\ y(1) = 1 \\ y'(1) = 2\,. \end{cases}$$

Compute the limit $\ell := \lim_{x\to+\infty} \mathrm{e}^{-x} y(x)$.

3. Consider in $[0, +\infty)$ the sequence of functions

$$f_n(x) = \sin\left(\frac{\pi}{2+(3x)^n}\right).$$

a) Compute the pointwise limit f of f_n on $[0, +\infty)$ as $n \to +\infty$.

b) Is it true that $f_n \to f$ in $C^0([0, +\infty))$ (with the sup-norm) as $n \to +\infty$?

c) Find the values of a such that $f_n \to f$ in $C^0([a, +\infty))$ (with the sup-norm) as $n \to +\infty$.

4. Let, for $x \in \mathbf{R}$, $f_\lambda(x) := \frac{\lambda}{(x^2+\lambda^2)\pi}$. Then

a) Prove that $f_\lambda \in L^1(\mathbf{R})$ and compute $\int_{\mathbf{R}} f_\lambda(x)\,dx$.

b) Compute the limit in $\mathcal{D}'(\mathbf{R})$ of f_λ as $\lambda \to 0^+$ (Hint: use the Lebesgue Theorem).