Master Program in Electronic Engineering

Advanced Mathematical Methods for Engineers

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1. Let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = \dfrac{x^2 y^3}{1+y^2} \\ y(x_0) = y_0. \end{cases}$$

a) Discuss local and global existence of solutions.

b) Draw the graph of the solutions, studying the monotonicity, the existence of maxima and minima, flex, asymptots and limits at the extrema of the domain.

2. Given the following ODE system

$$\begin{cases} x' = 2x + y \\ y' = -x + 2y \end{cases}$$

2.1) Solve the system.

2.2) Find the bounded solutions on $(-\infty, 0]$.

3. Consider, for $x \in (0, +\infty)$ the sequence of functions f_n:

$$f_n(x) = \left(\frac{\pi}{2} - \arctan(nx)\right)^2$$

a) Prove that $f_n \in L^1(0, +\infty)$ for every $n \in \mathbf{N}$.

b) Compute the pointwise limit f of f_n as $n \to \infty$.

c) Compute the $\lim_{n\to\infty} \int_0^{+\infty} f_n(x)\, dx$, justifying the computations.
Hint: it is usefull to use the fact that $(\arctan(x)+\arctan(1/x))' = 0$ on $(-\infty, 0) \cup (0, +\infty)$.

4. In the Hilbert space $H = L^2(0, 1)$ consider the closed subspace V of quadratic polynomials

$$V = \{at^2 + bt + c \,:\, a, b, c \in \mathbf{R}\}$$

compute the projection of f on V: $P_V f$, where $f(t) = t^3$.