Master Program in Electronic Engineering

# Advanced Mathematical Methods for Engineers

**June 27, 2023**

**1.** Let $(x_0, y_0) \in \mathbf{R} \times \mathbf{R}$, consider the following Cauchy Problem

$$\begin{cases} y'(x) = (y(x) - 1)\cos(x) \\ y(x_0) = y_0. \end{cases}$$

a) Discuss local and global existence and uniqueness of solutions, depending on $(x_0, y_0)$.

b) Find explicitly the solution in case $(x_0, y_0) = (0, -1)$ and draw its graph.

c) Prove the boundedness of the solution at point b).

**2.** Given the ODE equation

$$y'' + y = \sin(2x)$$

find the solutions $\bar{y}$ such that $\bar{y}(0) = 0$. Does the equation admit unbounded solutions? (Justify the answer).

**3.** Let, for $x > 0$ and $n \in \mathbf{N}$,

$$f_n(x) := \frac{e^{-x}}{1 + nx}.$$

Then

a) Find the pointwise limit $f$ of $f_n$ as $n$ tends to $+\infty$.

b) Prove that $f_n \in L^1(0, +\infty)$ for every $n$.

c) Compute, justifying the passages, the $\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)\, dx$ .

**4.** Let $h > 0$, and $\delta_h(x) := \frac{1}{h}\chi_{[-h/2,h/2]}(x)$, where

$$\chi_{[-h/2,h/2]}(x) := \begin{cases} 1 & \text{if } x \in [-h/2, h/2] \\ 0 & \text{otherwise.} \end{cases}$$

Prove that

a) $\delta_h \to \delta_0$ as $h \to 0$;

b) if $f(x) = ax + b$, $a$, $b \in \mathbf{R}$, then $\delta_h * f = f$ for every $h > 0$;

c) if $f(x) = ax^2 + bx + c$, $a$, $b$, $c \in \mathbf{R}$, then $\delta_h * f = f + a\frac{h^2}{12}$ for every $h > 0$.