Existence and long-time dynamics of a nonlocal Cahn-Hilliard-Navier-Stokes system with nonconstant mobility

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INTERFACES and Free Boundaries: ANALYSIS, Control and Simulation

Oberwolfach, 24 March - 30 March, 2013

joint preprint arXiv:1303.6446 with
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Supported by the FP7-IDEAS-ERC-StG Grant "EntroPhase" #256872

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  - ◇ well-posedness
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Modelling motivation

- A well-known model which describes the evolution of an incompressible isothermal mixture of two immiscible fluids is the so-called **model H** (cf. [Gurtin, Polignone, Viñals, '96], [Hohenberg, Halperin, '77], [H. Abels' seminar])
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  - an order parameter φ (concentration difference of the two components) is introduced to demarcate the two species and to indicate the location of the interface
  - mixing energy F is defined in terms of φ and its spatial gradient

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- The time evolution of φ is described by means of a convection-diffusion equation: typically, different variants of **Cahn-Hilliard** or Allen-Cahn or other types of dynamics are used (see [Anderson et al., '98], [Feng, '06])
- This parameter influences the (average) fluid velocity $\mathbf{u}$ through a capillarity force (called **Korteweg force**) proportional to $\mu\nabla\varphi$, where μ is the chemical potential (cf. [Jasnow, Viñals, '96])

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$$\varphi_t + \mathbf{u}\cdot\nabla\varphi = \mathrm{div}(m(\varphi)\nabla\mu), \qquad \mu = -\sigma\Delta\varphi + \frac{1}{\sigma}F'(\varphi)$$

where

- m denotes the non-constant mobility
- μ the chemical potential
- F the (density of) potential energy (logarithmic or double-well potential)
- $\mu\nabla\varphi$ is the so-called **Korteweg force**
- ν the viscosity and π the pressure
- $\sigma > 0$ **is related to the (diffuse) interface thickness**

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$$E(\varphi) = \int_\Omega \left(\frac{\sigma}{2} |\nabla \varphi(x)|^2 + \frac{F(\varphi(x))}{\sigma} \right) dx$$

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- (in the nonlocal case, cf. [Gajewski, Zacharias, '03], ..., [Colli, Frigeri, Grasselli, '12])

$$E(\varphi) = \frac{1}{4} \int_\Omega \int_\Omega J(x-y) \left(\varphi(x) - \varphi(y)\right)^2 \, dx \, dy + \int_\Omega \eta F(\varphi(x)) \, dx$$

  - $J : \mathbb{R}^d \to \mathbb{R}$ is a smooth even function, e.g. $J(x) = j_3 |x|^{-1}$ in 3D and $J(x) = -j_2 \log |x|$ in 2D
  - it is justified as a macroscopic limit of microscopic phase segregation models with particle conserving dynamics (cf. [Giacomin Lebowitz, '97&'98])

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where we denote

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Big changes in the model and in the analysis:

- the fourth order equation becames a second order equation $\Longrightarrow$ more chance to get separation property and uniqueness
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A philosophical question: is diffusion local or nonlocal?

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If we consider

$$\Delta u = \lim_{\epsilon \to 0} \frac{c_\epsilon}{|B_\epsilon(x)|} \int_{B_\epsilon(x)} (u(y) - u(x))\, dy \,,$$

the density at the point x compares itself with its values in a tiny surrounding ball. The difference between the surrounding average and the value at x, properly scaled is the "Laplacian".

If the set to which u compares itself is not shrunk to zero, the process is an integral diffusion. More generally, for a positive symmetric kernel, it can be

$$Lu(x) = \int J(x,y)(u(y) - u(x))\, dy \,."$$

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and the corresponding initial-boundary value problem (in $\Omega \times (0, T)$) is

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&\varphi_t + \mathbf{u} \cdot \nabla \varphi = \operatorname{div}(m(\varphi)\nabla \mu) \\
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&\frac{\partial \mu}{\partial n} = 0, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T) \\
&\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega
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where

- m denotes the non-constant **mobility**
- μ the chemical potential
- $(J * \varphi)(x) := \int_\Omega J(x-y)\varphi(y)\, dy$, $a(x) := \int_\Omega J(x-y)\, dy$, $x \in \Omega$ (**nonlocal operator**)
- F **the (density of) potential energy (logarithmic or double-well potential)**
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The non degenerate mobility: assumptions

(H1) $m \in C^{0,1}_{loc}(\mathbb{R})$ and there exist $m_1, m_2 > 0$ such that $m_1 \leq m(s) \leq m_2$ for all $s \in \mathbb{R}$;

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(H2) $J(\cdot - x) \in W^{1,1}(\Omega)$ for a.a $x \in \Omega$, $J(x) = J(-x)$, $a(x) := \int_\Omega J(x-y)dy \ge 0$ and

$$a^* := \sup_{x\in\Omega} \int_\Omega |J(x-y)|dy < \infty, \; b := \sup_{x\in\Omega} \int_\Omega |\nabla J(x-y)|dy < \infty;$$

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(H3) (quadratic perturbation of a strictly convex function) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \ge c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega;$$

(H4) There exist $c_1 > (a^* - a_*)/2$ ($a_* := \inf_{x\in\Omega} \int_\Omega J(x-y)dy$) and $c_2 \in \mathbb{R}$ such that

$$F(s) \ge c_1 s^2 - c_2, \qquad \forall s \in \mathbb{R};$$

(H5) (fulfilled by arbitrary polynomially growing potentials) There exist $c_3 > 0$, $c_4 \ge 0$ and $r \in (1,2]$ such that

$$|F'(s)|^r \le c_3 |F(s)| + c_4, \qquad \forall s \in \mathbb{R}$$

Definition 1: the non degenerate mobility – notion of weak solutions

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Then, a couple $[\mathbf{u}, \varphi]$ is a *weak solution* to the PDE system on $[0, T]$ if

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)_{div}) \cap L^2(0, T; H^1(\Omega)_{div}), \ \varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

$$\mathbf{u}_t \in L^{4/3}(0, T; H^1(\Omega)^*_{div}), \qquad \varphi_t \in L^{4/3}(0, T; H^1(\Omega)^*), \quad \text{if } d = 3,$$

$$\mathbf{u}_t \in L^{2-\gamma}(0, T; H^1(\Omega)^*_{div}), \qquad \varphi_t \in L^{2-\delta}(0, T; H^1(\Omega)^*) \ (\gamma, \delta \in (0, 1)), \quad \text{if } d = 2$$

$$\mu := a\varphi - J * \varphi + F'(\varphi) \in L^2(0, T; H^1(\Omega))$$

and the following variational formulation is satisfied for a.a. $t \in (0, T)$

$$\langle \varphi_t, \psi \rangle + (m(\varphi)\nabla\mu, \nabla\psi) = (\mathbf{u}\varphi, \nabla\psi), \qquad \forall \psi \in H^1(\Omega)$$

$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -(\varphi\nabla\mu, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle, \qquad \forall \mathbf{v} \in H^1(\Omega)_{div}$$

together with the initial conditions $\mathbf{u}(0) = \mathbf{u}_0$, $\varphi(0) = \varphi_0$ in Ω and where

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w}, \qquad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)_{div}$$

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$$\mathcal{E}(\mathbf{u}(t),\varphi(t)) + \int_0^t \left(\nu\|\nabla\mathbf{u}\|^2 + \|\sqrt{m(\varphi)}\nabla\mu\|^2\right)d\tau \le \mathcal{E}(\mathbf{u}_0,\varphi_0) + \int_0^t \langle \mathbf{h}(\tau),\mathbf{u}\rangle d\tau$$

for every $t>0$, where we have set

$$\mathcal{E}(\mathbf{u}(t),\varphi(t)) = \frac{1}{2}\|\mathbf{u}(t)\|^2 + \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(\varphi(x,t)-\varphi(y,t))^2 dxdy + \int_\Omega F(\varphi(t))\,dx$$

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Furthermore, assume that [(H4): $F(s) \geq c_1 s^2 - c_2$] is replaced by

(H7) (fulfilled by the classical double well) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exist $c_5>0$, $c_6>0$ and $p>2$ such that

$$F''(s) + a(x) \geq c_5|s|^{p-2} - c_6,\ \forall s \in \mathbb{R},\ \text{a.e. } x \in \Omega, \quad a(x) := \int_\Omega J(x-y)dy$$

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$$F''(s) + a(x) \geq c_5|s|^{p-2} - c_6, \ \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad a(x) := \int_\Omega J(x-y)dy$$

Then, for every $T > 0$ there exists a weak solution $[u, \varphi]$ satisfying

$$\varphi \in L^\infty(0, T; L^p(\Omega)),$$

$$\varphi_t \in L^2(0, T; H^1(\Omega)^*), \quad \text{if} \quad d = 2 \quad \text{or} \quad (d = 3 \text{ and } p \geq 3),$$

$$\mathbf{u}_t \in L^2(0, T; H^1(\Omega)^*_{div}), \quad \text{if} \quad d = 2$$

The proof follows the line of [Colli, Frigeri, Grasselli, '12]

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- any weak solution satisfies the **energy identity**

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}, \varphi) + \nu \|\nabla \mathbf{u}\|^2 + \|\sqrt{m(\varphi)}\nabla \mu\|^2 = \langle \mathbf{h}(t), \mathbf{u} \rangle, \qquad t > 0$$

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  In particular we have

$$\mathbf{u} \in C([0,\infty); L^2(\Omega)_{div}), \quad \varphi \in C([0,\infty); L^2(\Omega)), \quad \int_\Omega F(\varphi) \in C([0,\infty))$$

- If in addition $\mathbf{h} \in L^2_{tb}(0,\infty; H^1(\Omega)^*_{div})$, then any weak solution satisfies also the **dissipative estimate**

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) \leq \mathcal{E}(\mathbf{u}_0, \varphi_0)e^{-kt} + F(m_0)|\Omega| + K, \qquad \forall t \geq 0,$$

  where $m_0 = (\varphi_0, 1)$ and k, K are two positive constants which are independent of the initial data, with K depending on Ω, ν, J, F and $\|\mathbf{h}\|_{L^2_{tb}(0,\infty;H^1(\Omega)^*_{div})}$

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(D1) $m \in C^1([-1,1])$, $m \geq 0$ and that $m(s) = 0$ if and only if $s = -1$ or $s = 1$, $F \in C^2(-1,1)$ and

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(D2) $F = F_1 + F_2$, $F_2 \in C^2([-1,1])$ and there exists $a_2 > 4(a^* - a_* - b_2)$, where $b_2 = \min F_2''$ and $\varepsilon_0 > 0$ such that

$$F_1^{''}(s) \geq a_2, \qquad \forall s \in (-1, -1+\varepsilon_0] \cup [1-\varepsilon_0, 1)$$

(D3) There exists $\varepsilon_0 > 0$ such that $F_1^{''}$ is non-decreasing in $[1-\varepsilon_0, 1)$ and non-increasing in $(-1, -1+\varepsilon_0]$

(D4) There exists $c_0 > 0$ such that

$$F^{''}(s) + a(x) \geq c_0, \qquad \forall s \in (-1,1), \qquad \text{a.e. } x \in \Omega$$

Examples of m and F

It is easy to see that (D1)–(D4) are satisfied in the physically relevant case where the mobility and the double-well potential are given by

$$m(s) = k_1(1 - s^2), \qquad F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\big((1+s)\log(1+s) + (1-s)\log(1-s)\big)$$

where $0 < \theta < \theta_c$. Indeed, setting $F_1(s) := (\theta/2)\big((1+s)\log(1+s) + (1-s)\log(1-s)\big)$ and $F_2(s) = -(\theta_c/2)s^2$, then we have

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Another example is given by

$$m(s) = k(s)(1 - s^2)^m, \qquad F(s) = -k_2 s^2 + F_1(s)$$

where $k \in C^1([-1,1])$ such that $0 < k_3 \le k(s) \le k_4$ for all $s \in [-1,1]$, and F_1 is a $C^2(-1,1)$ convex function such that

$$F_1''(s) = \ell(s)(1 - s^2)^{-m}, \qquad \forall s \in (-1,1)$$

where $m \ge 1$ and $\ell \in C^1([-1,1])$

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Let $u_0 \in L^2(\Omega)_{div}$, $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. A couple $[u, \varphi]$ is a *weak solution* on $[0, T]$ corresponding to $[u_0, \varphi_0]$ if

- $\mathbf{u}$, φ satisfy

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)_{div}) \cap L^2(0, T; H^1(\Omega)_{div})$$
$$\mathbf{u}_t \in L^{4/3}(0, T; H^1(\Omega)^*_{div}) \text{ (if } \quad d = 3), \ \mathbf{u}_t \in L^2(0, T; H^1(\Omega)^*_{div}) \text{ (if } \quad d = 2)$$
$$\varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \ \varphi_t \in L^2(0, T; H^1(\Omega)^*)$$
$$\varphi \in L^\infty(Q_T), \qquad |\varphi(x, t)| \le 1 \quad \text{a.e. } (x, t) \in Q_T := \Omega \times (0, T)$$

- for every $\psi \in H^1(\Omega)$, every $\mathbf{v} \in H^1(\Omega)_{div}$ and for almost any $t \in (0, T)$ we have

$$\langle \varphi_t, \psi \rangle + \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla\psi + \int_\Omega m(\varphi) a \nabla\varphi \cdot \nabla\psi$$
$$+ \int_\Omega m(\varphi)(\varphi \nabla a - \nabla J * \varphi) \cdot \nabla\psi = (\mathbf{u}\varphi, \nabla\psi)$$
$$\langle \mathbf{u}_t, v \rangle + \nu(\nabla \mathbf{u}, \nabla v) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = ((a\varphi - J * \varphi)\nabla\varphi, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle$$
$$\mathbf{u}(0) = \mathbf{u}_0, \ \varphi(0) = \varphi_0$$

Theorem 2: the degenerate mobility – existence of solutions

Introduce the function $M \in C^2(-1,1)$ defined by $m(s)M''(s) = 1, M(0) = M'(0) = 0$

Assume (D1)–(D4), (H2). Let $\mathbf{h} \in L^2(0,T; H^1(\Omega)^*_{div})$, $\mathbf{u}_0 \in L^2(\Omega)_{div}$, $\varphi_0 \in L^\infty(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$

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In addition, if $d = 2$, the weak solution $z := [\mathbf{u}, \varphi]$ satisfies the *the energetic equality*

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \|\varphi\|^2\right) + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_\Omega am(\varphi)|\nabla\varphi|^2 + \nu\|\nabla\mathbf{u}\|^2$$
$$= \int_\Omega m(\varphi)(\nabla J * \varphi - \varphi\nabla a)\cdot\nabla\varphi + \int_\Omega (a\varphi - J*\varphi)\mathbf{u}\cdot\nabla\varphi + \langle \mathbf{h}, \mathbf{u}\rangle$$

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If $d = 3$ and if (H7) is satisfied with $p \ge 3$, z satisfies the following *energetic inequality*

$$\frac{1}{2}\left(\|\mathbf{u}(t)\|^2 + \|\varphi(t)\|^2\right) + \int_0^t\int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 + \int_0^t\int_\Omega am(\varphi)|\nabla\varphi|^2$$
$$+ \nu\int_0^t \|\nabla\mathbf{u}\|^2 \le \frac{1}{2}\left(\|\mathbf{u}_0\|^2 + \|\varphi_0\|^2\right) + \int_0^t\int_\Omega m(\varphi)\left(\nabla J * \varphi - \varphi\nabla a\right)\cdot\nabla\varphi$$
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$$\varphi_t,\ \mu\nabla\varphi \in L^{4/3}(0,T;H^1(\Omega)^*) \text{ and so } \mathbf{u}_t \in L^{4/3}(0,T;H^1(\Omega)^*_{div})$$

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- We pass to the limit as $\varepsilon \searrow 0$ obtaining the **weak formulation** stated in Theorem 2

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for all $t > 0$, where the **mass flux** $\mathcal{J}$ is such that

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Note that in this case it can be proved that **the sets** $\{x \in \Omega : \varphi(x,t) = 1\}$ **and** $\{x \in \Omega : \varphi(x,t) = -1\}$ **have both measure zero** for a.a. $t > 0$

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- The assumptions on φ_0 imply only the **less strict condition $|\overline{\varphi}_0| \leq 1$**
- This is due due to the **different weak solution formulation** with respect to the case of constant mobility
- Therefore, if F is bounded (e.g. F is the logarithmic potential) and at $t = 0$ the fluid is in a pure phase, i.e. $\varphi_0 = 1$ a.e. in Ω, and furthermore $\mathbf{u}_0 = \mathbf{u}(0)$ is given in $L^2(\Omega)_{div}$, then the couple

$$\mathbf{u} = \mathbf{u}(x,t), \qquad \varphi = \varphi(x,t) = 1, \qquad \text{a.e. in } \Omega, \quad \text{a.a. } t,$$

  where $\mathbf{u}$ is solution of the Navier-Stokes equations with non-slip boundary condition **explicitly satisfies the weak formulation**
- **This possibility is excluded in the model with constant mobility** since in such model the chemical potential μ (and hence $F'(\varphi)$) appears explicitly

The degenerate vs. the strongly degenerate mobility case

- **If $m'(1) \neq 0$ and $m'(-1) \neq 0$**, then both F and M (s.t. $m(s)M''(s) = 1$, $M(0) = M'(0) = 0$) are bounded in $[-1, 1] \Longrightarrow$ the conditions $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$ of Theorem 2 are satisfied by every initial datum φ_0 such that $|\varphi_0| \leq 1$ in $\Omega \Longrightarrow$ the **existence of pure phases is allowed**

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- **If** $m'(1) = m'(-1) = 0$ (in this case we say that m is **strongly degenerate**), then it can be proved that the conditions $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$ imply that the sets $\{x \in \Omega : \varphi_0(x) = 1\}$ and $\{x \in \Omega : \varphi_0(x) = -1\}$ have both measure zero $\Longrightarrow$ $|\overline{\varphi}_0| < 1$ and furthermore it can be seen that also the sets $\{x \in \Omega : \varphi(x, t) = 1\}$ and $\{x \in \Omega : \varphi(x, t) = -1\}$ have both measure zero for a.a. $t > 0 \Longrightarrow$ **pure phases are not allowed** (even on subsets of Ω of positive measure)

Theorem 4: The case of more regular chemical potential

(Cf. [Gajewski, Zacharias, '03])
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As a consequence, $z = [u, \varphi]$ now also satisfies the **Definition 1 of weak solutions**, the energy inequality and, for $d = 2$, the energy identity

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Write the weak formulation of approximated the Cahn-Hilliard equation as

$$\langle \varphi', \psi \rangle + \big(m_\varepsilon(\varphi)\nabla(F'_{1\varepsilon}(\varphi)), \nabla\psi\big) - \big(m_\varepsilon(\varphi)\nabla(J * \varphi), \nabla\psi\big) \qquad \text{(w-CH)}$$
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By means of *some technical arguments* and using the assumptions on F and, in particular, the condition $F'(\varphi_0) \in L^2(\Omega)$, we get

$$F'(\varphi) \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \Longrightarrow \mu \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$$

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Introduce the set $\mathcal{G}_{m_0}$ of all *weak solutions* (in the sense of **Definition 2**) corresponding to all initial data $z_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$, where the phase space $\mathcal{X}_{m_0}$ is the metric space defined by

$$\mathcal{X}_{m_0} := L^2(\Omega)_{div} \times \mathcal{Y}_{m_0}$$

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(D5) m, F satisfy (A1) and there exists $\alpha_0 > 0$ and $\rho \in [0,1)$ such that

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We point out that the existence of the global attractor is established **without the restriction $|\overline{\varphi}| < 1$ on the generalized semiflow**. In particular, this result does not require the separation property

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This estimate easily yields

$$d^2(z(t), 0) \leq d^2(z_0, 0)e^{-\eta t} + \frac{2C_3}{\eta} + |\overline{\varphi}_0|^2|\Omega|, \qquad \forall t \geq 0$$

where $d(z_2, z_1) := \|\mathbf{u}_2 - \mathbf{u}_1\| + \|\varphi_2 - \varphi_1\|$

Third part: The convective nonlocal Cahn-Hilliard equation with degenerate mobility

Assume that (D1)–(D4) are satisfied. Let $\mathbf{u} \in L^2_{loc}([0,\infty); H^1(\Omega)_{div} \cap L^\infty(\Omega)^d)$ be given and let $\mathbf{h} \in H^1(\Omega)^*_{div}$, $\varphi_0 \in L^\infty(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$

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Then, for every $T > 0$ there exists a weak solution φ to

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Furthermore, $\varphi \in L^\infty(0, T; L^p(\Omega))$, where $p \le 6$ for $d = 3$ and $2 \le p < \infty$ for $d = 2$. In addition, the following energy identity holds

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \int_\Omega m(\varphi) F''(\varphi) |\nabla\varphi|^2 + \int_\Omega a m(\varphi) |\nabla\varphi|^2 + \int_\Omega m(\varphi)\big(\varphi \nabla a - \nabla J * \varphi\big) \cdot \nabla\varphi = 0$$

for a.a. $t > 0$ and in $\mathcal{D}'(0,\infty)$

The convective nonlocal Cahn-Hilliard equation: uniqueness and attractor

Let the hypotheses of the previous Theorem be satisfied and the following conditions (cf. **(D5)**) are fulfilled for some $\alpha_0 > 0$ and $\rho \in [0, 1)$

$$m(s)F_1''(s) \geq \alpha_0 > 0 \qquad \forall s \in [-1, 1]$$
$$\rho F_1''(s) + F_2''(s) + a(x) \geq 0 \quad \forall s \in (-1, 1), \quad \text{for a.e. } x \in \Omega$$

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It is then immediate to check that the arguments used in the proofs of the previous results can be adapted to the present situation. Hence we have that: given $\mathbf{u} \in L^\infty(\Omega)^d$ independent of time, then, **the dynamical system $(\mathcal{Y}_{m_0}, S(t))$ possesses a connected global attractor**

Note that: up to our knowledge **uniqueness of solutions is an open issue** for the local case as well as for the complete nonlocal system including Navier-Stokes even in dimension two.

An idea of the proof of uniqueness

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Rewrite the Cahn-Hilliard equation as

$$\langle \varphi_t, \psi \rangle + \big(\nabla \Lambda(\cdot, \varphi), \nabla \psi\big) - \big(\Gamma(\varphi)\nabla a, \nabla \psi\big) + \big(m(\varphi)(\varphi \nabla a - \nabla J * \varphi), \nabla \psi\big) = \big(\mathbf{u}\varphi, \nabla \psi\big),$$

for all $\psi \in H^1(\Omega)$, where $\Lambda(x,s) := \Lambda_1(s) + \Lambda_2(s) + a(x)\Gamma(s)$ and

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for all $s \in [-1,1]$. Take the difference between the two identities, set $\varphi := \varphi_1 - \varphi_2$ and $\psi = \mathcal{N}\varphi$ (notice that $\overline{\varphi} = 0$):

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{N}^{1/2}\varphi\|^2 + \big(\Lambda(\varphi_2) - \Lambda(\varphi_1), \varphi\big) - \big((\Gamma(\varphi_2) - \Gamma(\varphi_1))\nabla a, \nabla \mathcal{N}\varphi\big)$$
$$+ \big((m(\varphi_2) - m(\varphi_1))(\varphi_2 \nabla a - \nabla J * \varphi_2) + m(\varphi_1)(\varphi \nabla a - \nabla J * \varphi), \nabla \mathcal{N}\varphi\big)$$
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On account of $m(s)F_1''(s) \geq \alpha_0 > 0$, $\rho F_1''(s) + F_2''(s) + a(x) \geq 0$, we find

$$\begin{aligned} \big(\Lambda(\cdot, \varphi_2) - \Lambda(\cdot, \varphi_1), \varphi\big) \geq & (1-\rho)\int_\Omega m(\theta\varphi_2 + (1-\theta)\varphi_1) F_1''(\theta\varphi_2 + (1-\theta)\varphi_1)\varphi^2 \\ & \geq (1-\rho)\alpha_0 \|\varphi\|^2 \end{aligned}$$

and the other terms can be estimated in order to apply Gronwall.

Conclusions

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We have proved in [Frigeri, Grasselli, E.R., preprint arXiv:1303.6446, 2013]

- Existence of solutions for the nonlocal 3D Navier-Stokes Cahn-Hilliard model with nondegenerate and with degenerate mobility
- Existence of the attractor in the 2D case
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There are still a lot of open problems like

- The case of non-smooth potentials like $F(\varphi) = I_{[-1,1]}(\varphi)$
- The case of unmatched densities (cf. [Abels, Depner, Garcke, 2013] for the local case) or of compressible fluids (cf. [Abels, Feireisl, 2008] for the local case)
- The non isothermal case (cf. [Eleuteri, E.R., Schimperna, work in progress] for the local case)
- ...

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/

Some comparisons with other results: local vs nonlocal

Results	Local CH	Nonlocal CH	Local CHNS	Nonlocal CHNS
Uniqueness	3D: True for non-degenerate mobility (e.g. [Elliott, '89, Novick Cohen, '9, [Elliott, Luckhaus, '91]) Open for degenerate mobility and singular potential	3D: True for constant mobility (e.g. [Colli, Krejčí, E.R., Sprekels, '04]) 3D: True for degenerate mobility and singular potential [Gajewski, Zacharias, '03, [Grasselli, Frigeri, E.R., '13]	2D: True for nondegenerate mobility [Abels, '09, Boyer, '99] Open for degenerate mobility and singular potential	Open even in 2D Open even in 2D
Separation	2D: True with logaritmich potential and constant mobility [Miranville, Zelik, '04] , 3D: Open for the logarithmic potential	3D: true for degenerate mobility and singular potential [Londen, Petzeltovà, '11]	Open	3D: true for degenerate mobility and singular potential