Degenerating PDE system for phase transitions and damage

E. Rocca

Università degli Studi di Milano

The 9th AIMS Conference on Dynamical Systems,
Differential Equations and Applications, Orlando, Florida, USA

SS2 Nonlinear Evolution PDEs and Interfaces in Applied Sciences

joint work with Riccarda Rossi (University of Brescia)

Supported by the FP7-IDEAS-ERC-StG Grant "EntroPhase"

Contents

Part 1. Introduction of the problems and deduction of the PDE system via modelling

Part 2. Our most recent results:

- joint with Riccarda Rossi [preprint arXiv:1205.3578v1 (2012)]: weak solvability of the 3D degenerating PDE system

Mathematical problems arising from Thermomechanics

Mathematical problems arising from Thermomechanics

- **Damage** phenomena:

Mathematical problems arising from Thermomechanics

- **Damage** phenomena:
  - **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
    - the evolution of the displacement variables
    - the temperature
    - the damage parameter χ (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, in damage models, while $0 < \chi < 1$ corresponds to *partial damage*)

Mathematical problems arising from Thermomechanics

- **Damage** phenomena:
  - **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
    - the evolution of the displacement variables
    - the temperature
    - the damage parameter χ (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, in damage models, while $0 < \chi < 1$ corresponds to *partial damage*)
- **Phase transitions** in thermoviscoelastic materials

Mathematical problems arising from Thermomechanics

- **Damage** phenomena:
  - **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
    - the evolution of the displacement variables
    - the temperature
    - the damage parameter χ (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, in damage models, while $0 < \chi < 1$ corresponds to *partial damage*)
- **Phase transitions** in thermoviscoelastic materials
  - **aim:** introduce a model where we have the full elastic contribution of

$$(1 - \chi)\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})$$

only in the non-viscous phase, i.e. when $\chi = 0$, while it is null in the viscous one, i.e. when $\chi = 1$:

- χ is the order parameter, standing for the local proportion of the liquid phase
- $\chi = 0$ stands for the solid phase,
- $\chi = 1$ for the liquid one
- $0 < \chi < 1$ in the so-called *mushy regions*

Mathematical problems arising from Thermomechanics

- **Damage** phenomena:
  - **aim:** deal with diffuse interface models in thermoviscoelasticity accounting for
    - the evolution of the displacement variables
    - the temperature
    - the damage parameter χ (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, in damage models, while $0 < \chi < 1$ corresponds to *partial damage*)
- **Phase transitions** in thermoviscoelastic materials
  - **aim:** introduce a model where we have the full elastic contribution of

$$(1 - \chi)\varepsilon(\mathbf{u})\mathrm{R}_e\varepsilon(\mathbf{u})$$

  only in the non-viscous phase, i.e. when $\chi = 0$, while it is null in the viscous one, i.e. when $\chi = 1$:
    - χ is the order parameter, standing for the local proportion of the liquid phase
    - $\chi = 0$ stands for the solid phase,
    - $\chi = 1$ for the liquid one
    - $0 < \chi < 1$ in the so-called *mushy regions*

where the momentum equation contains χ-dependent elliptic operators, which may degenerate at the *pure phases* 0 and 1

The scope

The analysis of the initial boundary-value problem for the following PDE system:

$$\begin{aligned}
&c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \,\mathrm{div}\, \mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g \\
&\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \\
&\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta
\end{aligned}$$

which describes a thermoviscoelastic system in a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ during a time interval $[0, T]$

The scope

The analysis of the initial boundary-value problem for the following PDE system:

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta) = g$$
$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

which describes a thermoviscoelastic system in a reference domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ during a time interval $[0, T]$

- ϑ is the absolute temperature of the system
- $\mathbf{u}$ the vector of *small displacements*
- χ is the order parameter, standing for the local proportion of one of the two phases in *phase transitions* ($\chi = 0$: solid phase and $\chi = 1$: liquid phase, and $0 < \chi < 1$ in the so-called *mushy regions*) $\Longrightarrow a(\chi) = \chi,\ b(\chi) = 1 - \chi$
- χ is the damage parameter, assessing the soundness of the material in *damage* (for the completely *damaged* $\chi = 0$ and the *undamaged* state $\chi = 1$, respectively, while $0 < \chi < 1$: *partial damage*) $\Longrightarrow a(\chi) = b(\chi) = \chi$

Deal with the possible degeneracy in the momentum equation

Deal with the possible degeneracy in the momentum equation

Main aim: We shall let a and b vanish at the threshold values 0 and 1, not enforce separation of χ from the threshold values 0 and 1, and accordingly we will allow for general initial configurations of χ

Deal with the possible degeneracy in the momentum equation

Main aim: We shall let a and b vanish at the threshold values 0 and 1, not enforce separation of χ from the threshold values 0 and 1, and accordingly we will allow for general initial configurations of χ

$\Longrightarrow$ It is not to be expected that either of the coefficients $a(\chi)$ and $b(\chi)$ stay away from 0: elliptic degeneracy of the displacement equation

$$\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

Deal with the possible degeneracy in the momentum equation

Main aim: We shall let a and b vanish at the threshold values 0 and 1, not enforce separation of χ from the threshold values 0 and 1, and accordingly we will allow for general initial configurations of χ

$\Longrightarrow$ It is not to be expected that either of the coefficients $a(\chi)$ and $b(\chi)$ stay away from 0: elliptic degeneracy of the displacement equation

$$\mathbf{u}_{tt} - \mathrm{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

$\Longrightarrow$ We shall approximate the system with a non-degenerating one, where we replace the momentum equation with

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f} \qquad \text{for } \delta > 0$$

The first results and the new goal

The first results and the new goal

[FIRST RESULT.] **Local in time well-posedness** for a suitable formulation of the reversible problem ($\mu = 0$ and $\rho = 0$) using in

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 \,.$$

the *small perturbations assumption* in the 3D (in space) setting [J. DIFFERENTIAL EQUATIONS, 2008]

The first results and the new goal

[First result.] **Local in time well-posedness** for a suitable formulation of the reversible problem ($\mu = 0$ and $\rho = 0$) using in

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \,\mathrm{div}\, \mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 \,.$$

the *small perturbations assumption* in the 3D (in space) setting [J. Differential Equations, 2008]

[Second result.] **Global well-posedness** in the 1D case without *small perturbations assumption* [Appl. Math., Special Volume (2008)]

The first results and the new goal

[First result.] **Local in time well-posedness** for a suitable formulation of the reversible problem ($\mu = 0$ and $\rho = 0$) using in

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 .$$

the *small perturbations assumption* in the 3D (in space) setting [J. Differential Equations, 2008]

[Second result.] **Global well-posedness** in the 1D case without *small perturbations assumption* [Appl. Math., Special Volume (2008)]

Note: in both these results we assumed χ_0 separated from the thresholds 0 and 1 and we prove (exploiting a sufficient coercivity condition on W at the thresholds 0 and 1) that the solution χ during the evolution continues to stay separated from 0 and 1 $\Longrightarrow$ **prevent degeneracy** (the operators are uniformly elliptic)

The first results and the new goal

[First result.] **Local in time well-posedness** for a suitable formulation of the reversible problem ($\mu = 0$ and $\rho = 0$) using in

$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2 .$$

the *small perturbations assumption* in the 3D (in space) setting [J. Differential Equations, 2008]

[Second result.] **Global well-posedness** in the 1D case without *small perturbations assumption* [Appl. Math., Special Volume (2008)]

Note: in both these results we assumed χ_0 separated from the thresholds 0 and 1 and we prove (exploiting a sufficient coercivity condition on W at the thresholds 0 and 1) that the solution χ during the evolution continues to stay separated from 0 and 1 $\Longrightarrow$ **prevent degeneracy** (the operators are uniformly elliptic)

The goal [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]: to establish a **global existence result in 3D** using a suitable notion of solution and without enforcing the separation property, i.e. **allowing for degeneracy**

The model

Free energy and Dissipation

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of the UMI 13, Springer-Verlag, Berlin, 2012]

Free energy and Dissipation

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of the UMI 13, Springer-Verlag, Berlin, 2012]

The free-energy $\mathcal{F}$:

$$\mathcal{F} = \int_\Omega \left(f(\vartheta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{1}{p}|\nabla\chi|^p + W(\chi) + \rho\vartheta \mathrm{tr}(\varepsilon(\mathbf{u})) - \vartheta\chi \right) \mathrm{d}x$$

- f is a concave function, $\rho \in \mathbb{R}$ a thermal expansion coefficient
- $b \in C^2(\mathbb{R}; [0, +\infty))$, e.g., $b(\chi) = 1 - \chi$ in phase transitions, $b(\chi) = \chi$ in damage
- $p > d$: we need the embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0, 1]$

Free energy and Dissipation

cf. [M. Frémond, Phase Change in Mechanics, Lecture Notes of the UMI 13, Springer-Verlag, Berlin, 2012]

The free-energy $\mathcal{F}$:

$$\mathcal{F} = \int_\Omega \left(f(\vartheta) + b(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \frac{1}{p}|\nabla\chi|^p + W(\chi) + \rho\vartheta \mathrm{tr}(\varepsilon(\mathbf{u})) - \vartheta\chi \right) \mathrm{d}x$$

- f is a concave function, $\rho \in \mathbb{R}$ a thermal expansion coefficient
- $b \in C^2(\mathbb{R}; [0, +\infty))$, e.g., $b(\chi) = 1 - \chi$ in phase transitions, $b(\chi) = \chi$ in damage
- $p > d$: we need the embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$
- $W = \widehat{\beta} + \widehat{\gamma}$, $\widehat{\gamma} \in C^2(\mathbb{R})$, $\widehat{\beta}$ proper, convex, l.s.c., $\overline{\mathrm{dom}(\widehat{\beta})} = [0, 1]$

The pseudo-potential $\mathcal{P}$:

$$\mathcal{P} = \frac{k(\vartheta)}{2}|\nabla\vartheta|^2 + \frac{1}{2}|\chi_t|^2 + a(\chi)\frac{|\varepsilon(\mathbf{u}_t)|^2}{2} + \mu I_{(-\infty,0]}(\chi_t)$$

- k the heat conductivity: coupled conditions with the specific heat $\mathrm{c}(\vartheta) = f(\vartheta) - \vartheta f'(\vartheta)$
- $a \in C^1(\mathbb{R}; [0, +\infty))$, e.g., $a(\chi) = \chi$
- $\mu = 0$: reversible case, $\mu = 1$: irreversible case

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)}\right) \quad \text{becomes}$$

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}}$$

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div}\sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)}\right) \quad \text{becomes}$$

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}}$$

The phase evolution (standard principle of virtual powers)

$$B - \operatorname{div}\mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi}\right) \quad \text{becomes}$$

$$\boxed{\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta}$$

The modelling

The momentum equation

$$\mathbf{u}_{tt} - \operatorname{div} \sigma = \mathbf{f} \quad \left(\sigma = \sigma^{nd} + \sigma^{d} = \frac{\partial \mathcal{F}}{\partial \varepsilon(\mathbf{u})} + \frac{\partial \mathcal{P}}{\partial \varepsilon(\mathbf{u}_t)} \right) \quad \text{becomes}$$

$$\boxed{\mathbf{u}_{tt} - \operatorname{div}(a(\chi)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}}$$

The phase evolution (standard principle of virtual powers)

$$B - \operatorname{div} \mathbf{H} = 0 \quad \left(B = \frac{\partial \mathcal{F}}{\partial \chi} + \frac{\partial \mathcal{P}}{\partial \chi_t}, \mathbf{H} = \frac{\partial \mathcal{F}}{\partial \nabla \chi} \right) \quad \text{becomes}$$

$$\boxed{\chi_t + \mu \partial I_{(-\infty,0]}(\chi_t) - \Delta_p \chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta}$$

The internal energy balance

$$e_t + \operatorname{div} \mathbf{q} = g + \sigma : \varepsilon(\mathbf{u}_t) + B\chi_t + \mathbf{H} \cdot \nabla \chi_t \quad \left(e = \mathcal{F} - \vartheta \frac{\partial \mathcal{F}}{\partial \vartheta}, \quad \mathbf{q} = \frac{\partial \mathcal{P}}{\partial \nabla \vartheta} \right)$$

becomes

$$\boxed{\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2}$$

The analysis

Main mathematical difficulties

Main mathematical difficulties

- Our main aim is **to handle the elliptic degeneracy** of the momentum equation: we replace it by

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$

and we want let $\delta \searrow 0$

Main mathematical difficulties

- Our main aim is **to handle the elliptic degeneracy** of the momentum equation: we replace it by
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$
and we want let $\delta \searrow 0$
- We should also treat the *nonlinear coupling* between the single equations: the heat equation
$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2$$
and the phase equation
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta \qquad \text{(Phase)}$$

Main mathematical difficulties

- Our main aim is **to handle the elliptic degeneracy** of the momentum equation: we replace it by
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$
and we want let $\delta \searrow 0$
- We should also treat the *nonlinear coupling* between the single equations: the heat equation
$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2$$
and the phase equation
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta \qquad \text{(Phase)}$$
- A major difficulty stems from the simultaneous presence in (Phase) of $\partial I_{(-\infty,0]}(\chi_t)$, $-\Delta_p\chi$, and $W'(\chi)$. However $-\Delta_p\chi$ is necessary in order to estimate $-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}$

Main mathematical difficulties

- Our main aim is **to handle the elliptic degeneracy** of the momentum equation: we replace it by
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$
and we want let $\delta \searrow 0$

- We should also treat the *nonlinear coupling* between the single equations: the heat equation
$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2$$
and the phase equation
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta \qquad \text{(Phase)}$$

- A major difficulty stems from the simultaneous presence in (Phase) of $\partial I_{(-\infty,0]}(\chi_t)$, $-\Delta_p\chi$, and $W'(\chi)$. However $-\Delta_p\chi$ is necessary in order to estimate $-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}$

- We consider a suitable weak formulation of (Phase) consisting of a one-sided variational inequality + an energy inequality $\Longrightarrow$ **generalized principle of virtual powers**

Main mathematical difficulties

- Our main aim is **to handle the elliptic degeneracy** of the momentum equation: we replace it by
$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}$$
and we want let $\delta \searrow 0$

- We should also treat the *nonlinear coupling* between the single equations: the heat equation
$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta) = g + |\chi_t|^2 + a(\chi)|\varepsilon(\mathbf{u}_t)|^2$$
and the phase equation
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta \qquad \text{(Phase)}$$

- A major difficulty stems from the simultaneous presence in (Phase) of $\partial I_{(-\infty,0]}(\chi_t)$, $-\Delta_p\chi$, and $W'(\chi)$. However $-\Delta_p\chi$ is necessary in order to estimate $-b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2}$

- We consider a suitable weak formulation of (Phase) consisting of a one-sided variational inequality + an energy inequality $\Longrightarrow$ **generalized principle of virtual powers**

- In a first approach, we take the *small perturbation assumption* and deal with
$$c(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta) = g$$

Main new results

The main ideas to handle nonlinearities and degeneracy

The main ideas to handle nonlinearities and degeneracy

Introduce a concept of weak solution satisfying

a *generalization of the principle of virtual powers* inspired by

The main ideas to handle nonlinearities and degeneracy

Introduce a concept of weak solution satisfying

a *generalization of the principle of virtual powers* inspired by

1. the notion of *energetic solution* - A. Mielke and co-authors ([Bouchitté, Mielke, Roubíček, ZAMP. Angew. Math. Phys. (2009) and [Mielke, Roubíček, Zeman, Comput. Methods Appl. Mech. Engrg. (2010)]) for rate-independent processes for damaging phenomena and
2. a notion of *weak solution* introduced by [Heinemann, Kraus, WIAS preprint 1569 and WIAS preprint 1520, to appear on Adv. Math. Sci. Appl. (2010)] for non-degenerating isothermal diffuse interface models for phase separation and damage

Main results

Main results

- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0$$

Main results

- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0$$

- **[Thm. 1]** Existence of solutions to the non-degenerating system $\delta > 0$ in the *reversible* case, i.e. with $\mu = 0$ in

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

Main results

- We replace the momentum equation with a non-degenerating one
$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0$$
- **[Thm. 1]** Existence of solutions to the non-degenerating system $\delta > 0$ in the *reversible* case, i.e. with $\mu = 0$ in
$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$
- **[Thm. 2]** Existence of weak solutions to the non-degenerating system $\delta > 0$ in the *irreversible* case ($\mu = 1$) consisting of a *one-sided* variational inequality and of an energy inequality

Main results

- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \operatorname{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0$$

- **[Thm. 1]** Existence of solutions to the non-degenerating system $\delta > 0$ in the *reversible* case, i.e. with $\mu = 0$ in

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

- **[Thm. 2]** Existence of weak solutions to the non-degenerating system $\delta > 0$ in the *irreversible* case ($\mu = 1$) consisting of a *one-sided* variational inequality and of an energy inequality
- **[Thm. 3]** For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, 2009] and [Mielke, Roubíček, Zeman, 2011] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation

Main results

- We replace the momentum equation with a non-degenerating one

$$\mathbf{u}_{tt} - \mathrm{div}((a(\chi) + \delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u}) - \rho\vartheta\mathbf{1}) = \mathbf{f}, \quad \delta > 0$$

- **[Thm. 1]** Existence of solutions to the non-degenerating system $\delta > 0$ in the *reversible* case, i.e. with $\mu = 0$ in

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

- **[Thm. 2]** Existence of weak solutions to the non-degenerating system $\delta > 0$ in the *irreversible* case ($\mu = 1$) consisting of a *one-sided* variational inequality and of an energy inequality
- **[Thm. 3]** For the analysis of the degenerate limit $\delta \searrow 0$ we have carefully adapted techniques from [Bouchitté, Mielke, Roubíček, 2009] and [Mielke, Roubíček, Zeman, 2011] to the case of a *rate-dependent* equation for χ, also coupled with the temperature equation

- It seems to us that *both* the coefficients need to be truncated when taking the degenerate limit in the momentum equation: the truncation in front of $\varepsilon(\mathbf{u}_t)$ allows us to deal with the *main part* of the elliptic operator, but, in order to pass to the limit in the quadratic term on the right-hand side of χ-eq., we will also need to truncate the coefficient of $\varepsilon(\mathbf{u})$

Energy vs Enthalpy

In order to deal with the low regularity of ϑ, rewrite the internal energy equation

$$\mathsf{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta \operatorname{div} \mathbf{u}_t - \operatorname{div}(k(\vartheta)\nabla\vartheta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \rho\Theta(w) \operatorname{div} \mathbf{u}_t - \operatorname{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\vartheta) := \int_0^\vartheta \mathsf{c}(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{\mathsf{c}(\Theta(w))}$$

Energy vs Enthalpy

In order to deal with the low regularity of ϑ, rewrite the internal energy equation

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \rho\Theta(w)\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\vartheta) := \int_0^\vartheta \mathrm{c}(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{\mathrm{c}(\Theta(w))}$$

We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\,\sigma_1 \geq \sigma > \frac{2d}{d+2}$: $\quad c_0(1+\vartheta)^{\sigma-1} \leq \mathrm{c}(\vartheta) \leq c_1(1+\vartheta)^{\sigma_1-1} \Longrightarrow h$ is strictly increasing

Energy vs Enthalpy

In order to deal with the low regularity of ϑ, rewrite the internal energy equation

$$\mathrm{c}(\vartheta)\vartheta_t + \chi_t\vartheta - \rho\vartheta\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(k(\vartheta)\nabla\vartheta) = g$$

as the enthalpy equation

$$w_t + \chi_t\Theta(w) - \rho\Theta(w)\,\mathrm{div}\,\mathbf{u}_t - \mathrm{div}(K(w)\nabla w) = g \quad \text{where}$$

$$w = h(\vartheta) := \int_0^\vartheta \mathrm{c}(s)\,\mathrm{d}s, \quad \Theta(w) := \begin{cases} h^{-1}(w) & \text{if } w \geq 0, \\ 0 & \text{if } w < 0, \end{cases} \quad K(w) := \frac{k(\Theta(w))}{\mathrm{c}(\Theta(w))}$$

We assume that

- $\mathrm{c} \in C^0([0,+\infty);[0,+\infty))$
- $\exists\,\sigma_1 \geq \sigma > \frac{2d}{d+2}\ :\quad c_0(1+\vartheta)^{\sigma-1} \leq \mathrm{c}(\vartheta) \leq c_1(1+\vartheta)^{\sigma_1-1} \implies h$ is strictly increasing

Assume moreover

[If $\rho = 0$:] the function $k : [0,+\infty) \to [0,+\infty)$ is continuous, and

$$\exists\, c_2,\ c_3 > 0\ \ \forall\vartheta \in [0,+\infty)\ :\quad c_2\mathrm{c}(\vartheta) \leq k(\vartheta) \leq c_3(\mathrm{c}(\vartheta)+1)$$

[If $\rho \neq 0$:] $\exists\, c_\rho > 0\, \exists\, q \geq \frac{d+2}{2d}\ :\ K(w) = c_\rho\left(|w|^{2q}+1\right) \quad \forall w \in [0,+\infty)$

The non-degenerate case

The approximating non-degenerate Problem [P_δ]

Given $\delta > 0$, $\mu \in \{0,1\}$, find (measurable) functions

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$
$$\mathbf{u} \in H^1(0,T;H^2(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}(0,T;H^1_0(\Omega)) \cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))$$
$$\chi \in L^\infty(0,T;W^{1,p}(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

for every $1 \le r < \frac{d+2}{d+1}$, fulfilling the initial conditions

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}_t(0,x) = \mathbf{v}_0(x) \qquad \text{for a.e. } x \in \Omega$$
$$\chi(0,x) = \chi_0(x) \qquad \text{for a.e. } x \in \Omega$$

the equations (for every $\varphi \in \mathrm{C}^0([0,T];W^{1,r'}(\Omega)) \cap W^{1,r'}(0,T;L^{r'}(\Omega))$ and $t \in (0,T]$)

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x$$
$$- \rho \int_0^t \int_\Omega \mathrm{div}\mathbf{u}_t \Theta(w)\varphi \,\mathrm{d}x + \int_0^t \int_\Omega K(w)\nabla w \nabla\varphi \,\mathrm{d}x = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\,\mathrm{d}x$$
$$\mathbf{u}_{tt} - \mathrm{div}\left((a(\chi)+\delta)\varepsilon(\mathbf{u}_t) + b(\chi)\varepsilon(\mathbf{u})\right) - \rho\nabla\Theta(w) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion (in $W^{1,p}(\Omega)^*$ and a.e. in $(0,T)$)

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + \beta(\chi) + \gamma(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

Theorem 1 [The reversible case $\mu = 0$]

Theorem 1 [The reversible case $\mu = 0$]

Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$

$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$

$$\mathbf{u}_0 \in H_0^2(\Omega), \quad \mathbf{v}_0 \in H_0^1(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

Theorem 1 [The reversible case $\mu = 0$]

Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0, T; L^2(\Omega)), \quad g \in L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)')$$
$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$
$$\mathbf{u}_0 \in H_0^2(\Omega), \quad \mathbf{v}_0 \in H_0^1(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

Then,

1. Problem [P_δ] admits a solution $(w, \mathbf{u}, \chi)$, such that there exists

$$\xi \in L^2(0, T; L^2(\Omega)), \; \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (0,T) \; :$$
$$\chi_t - \Delta_p \chi + \xi + \gamma(\chi) = -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \qquad \text{a.e. in } \; \Omega \times (0,T)$$

Theorem 1 [The reversible case $\mu = 0$]

Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$
$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$
$$\mathbf{u}_0 \in H_0^2(\Omega), \quad \mathbf{v}_0 \in H_0^1(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

Then,

1. Problem $[\mathrm{P}_\delta]$ admits a solution $(w, \mathbf{u}, \chi)$, such that there exists

$$\xi \in L^2(0,T;L^2(\Omega)), \ \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (0,T) \ :$$
$$\chi_t - \Delta_p \chi + \xi + \gamma(\chi) = -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \qquad \text{a.e. in } \Omega \times (0,T)$$

2. Suppose that $g(x,t) \geq 0$ a.e. Then, $w \geq 0$ a.e., hence $\vartheta(x,t) := \Theta(w(x,t)) \geq 0$ a.e.

Theorem 1 [The reversible case $\mu = 0$]

Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$
$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$
$$\mathbf{u}_0 \in H_0^2(\Omega), \quad \mathbf{v}_0 \in H_0^1(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

Then,

1. Problem $[\mathrm{P}_\delta]$ admits a solution $(w, \mathbf{u}, \chi)$, such that there exists
$$\xi \in L^2(0,T;L^2(\Omega)),\ \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (0,T)\ :$$
$$\chi_t - \Delta_p\chi + \xi + \gamma(\chi) = -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \qquad \text{a.e. in } \ \Omega \times (0,T)$$
2. Suppose that $g(x,t) \geq 0$ a.e. Then, $w \geq 0$ a.e., hence $\vartheta(x,t) := \Theta(w(x,t)) \geq 0$ a.e.
3. In case $\rho \neq 0$, $w_0 \in L^2(\Omega)$, and $K(w) = c_\rho\left(|w|^{2q} + 1\right)$, $q \geq (d+2)/2d$.

Theorem 1 [The reversible case $\mu = 0$]

Let $\mu = 0$ and $\rho = 0$, assume the previous Hypotheses and the conditions:

$$\mathbf{f} \in L^2(0,T;L^2(\Omega)), \quad g \in L^1(0,T;L^1(\Omega)) \cap L^2(0,T;H^1(\Omega)')$$

$$\vartheta_0 \in L^{\sigma_1}(\Omega) \quad \text{whence} \quad w_0 := h(\vartheta_0) \in L^1(\Omega)$$

$$\mathbf{u}_0 \in H_0^2(\Omega), \quad \mathbf{v}_0 \in H_0^1(\Omega) \quad \chi_0 \in \mathrm{dom}(\Delta_p), \quad \widehat{\beta}(\chi_0) \in L^1(\Omega)$$

Then,

1. Problem $[\mathrm{P}_\delta]$ admits a solution $(w, \mathbf{u}, \chi)$, such that there exists

   $$\xi \in L^2(0,T;L^2(\Omega)), \ \xi(x,t) \in \beta(\chi(x,t)) \text{ for a.e. } (x,t) \in \Omega \times (0,T) \ :$$

   $$\chi_t - \Delta_p \chi + \xi + \gamma(\chi) = -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \qquad \text{a.e. in } \ \Omega \times (0,T)$$

2. Suppose that $g(x,t) \geq 0$ a.e. Then, $w \geq 0$ a.e., hence $\vartheta(x,t) := \Theta(w(x,t)) \geq 0$ a.e.
3. In case $\rho \neq 0$, $w_0 \in L^2(\Omega)$, and $K(w) = c_\rho \left(|w|^{2q} + 1\right)$, $q \geq (d+2)/2d$. Then, w has the further regularity

   $$w \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap W^{1,r(q)}((0,T);W^{2,-s(q)}(\Omega))$$

Theorem 2 [The irreversible case $\mu = 1$]

Let $\mu = 1$, $\rho = 0$, and take the previous assumptions with $\widehat{\beta} = I_{[0,+\infty)}$. Then,

[1.] Problem [P_δ] admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \varphi + \xi \varphi + \gamma(\chi)\varphi + b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} \varphi - \Theta(w)\varphi \geq 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T; L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{W^{1,p}(\Omega)} \leq 0 \ \ \forall \varphi \in W^{1,p}_+(\Omega), \text{ a.e. } t \in (0,T)$$

Theorem 2 [The irreversible case $\mu = 1$]

Let $\mu = 1$, $\rho = 0$, and take the previous assumptions with $\widehat{\beta} = I_{[0,+\infty)}$. Then,

[1.] Problem [P_δ] admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \varphi + \xi \varphi + \gamma(\chi)\varphi + b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} \varphi - \Theta(w)\varphi \geq 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T;L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{W^{1,p}(\Omega)} \leq 0 \ \ \forall \varphi \in W^{1,p}_+(\Omega), \text{ a.e. } t \in (0,T)$$

and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \leq t$:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{p} |\nabla \chi(t)|^p + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{p} |\nabla \chi(s)|^p + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

Theorem 2 [The irreversible case $\mu = 1$]

Let $\mu = 1$, $\rho = 0$, and take the previous assumptions with $\widehat{\beta} = I_{[0,+\infty)}$. Then,

[1.] Problem $[\mathrm{P}_\delta]$ admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \varphi + \xi \varphi + \gamma(\chi)\varphi + b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} \varphi - \Theta(w)\varphi \geq 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T; L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{W^{1,p}(\Omega)} \leq 0 \ \ \forall \varphi \in W^{1,p}_+(\Omega), \text{ a.e. } t \in (0,T)$$

and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \leq t$:

$$\begin{aligned} &\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x \,\mathrm{d}r + \frac{1}{p} |\nabla \chi(t)|^p + \int_\Omega W(\chi(t)) \,\mathrm{d}x \\ &\leq \frac{1}{p} |\nabla \chi(s)|^p + \int_\Omega W(\chi(s)) \,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \,\mathrm{d}x \,\mathrm{d}r \end{aligned}$$

[2.] Suppose in addition that $g(x,t) \geq 0$, $\vartheta_0 > \underline{\vartheta}_0 \geq 0$ a.e. Then $\vartheta(x,t) := \Theta(w(x,t)) \geq \underline{\vartheta}_0 \geq 0$ a.e.

Theorem 2 [The irreversible case $\mu = 1$]

Let $\mu = 1$, $\rho = 0$, and take the previous assumptions with $\widehat{\beta} = I_{[0,+\infty)}$. Then,

[1.] Problem $[P_\delta]$ admits a *weak solution* $(w, \mathbf{u}, \chi)$, which, beside fulfilling the enthalpy and momentum equations, satisfies $\chi_t(x,t) \leq 0$ for almost all $t \in (0,T)$, and $(\forall \varphi \in L^p(0,T; W^{1,p}_-(\Omega)) \cap L^\infty(Q))$ the *one-sided* inequality

$$\int_0^T \int_\Omega \chi_t \varphi + |\nabla \chi|^{p-2} \nabla \chi \cdot \nabla \varphi + \xi \varphi + \gamma(\chi)\varphi + b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} \varphi - \Theta(w)\varphi \geq 0$$

with $\xi \in \partial I_{[0,+\infty)}(\chi)$ in the following sense:

$$\xi \in L^1(0,T; L^1(\Omega)), \ \langle \xi(t), \varphi - \chi(t) \rangle_{W^{1,p}(\Omega)} \leq 0 \ \ \forall \varphi \in W^{1,p}_+(\Omega), \text{ a.e. } t \in (0,T)$$

and the energy inequality for all $t \in (0,T]$, for $s = 0$, and for almost all $0 < s \leq t$:

$$\int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x \,\mathrm{d}r + \frac{1}{p} |\nabla \chi(t)|^p + \int_\Omega W(\chi(t)) \,\mathrm{d}x$$
$$\leq \frac{1}{p} |\nabla \chi(s)|^p + \int_\Omega W(\chi(s)) \,\mathrm{d}x + \int_s^t \int_\Omega \chi_t \left(-b'(\chi) \frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x \,\mathrm{d}r$$

[2.] Suppose in addition that $g(x,t) \geq 0$, $\vartheta_0 > \underline{\vartheta}_0 \geq 0$ a.e. Then $\vartheta(x,t) := \Theta(w(x,t)) \geq \underline{\vartheta}_0 \geq 0$ a.e.

[3.] In case $\rho \neq 0$ an analogous statement to the reversible case holds true

Generalized principle of virtual powers vs classical phase inclusion

Generalized principle of virtual powers vs classical phase inclusion

- Any *weak solution* $(w, \mathbf{u}, \chi)$ fulfills the total energy inequality for all $t \in (0, T]$, for $s = 0$, and for almost all $0 < s \le t$

$$\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2 \,\mathrm{d}x + \int_s^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x + \int_s^t (\chi + \delta)|\varepsilon(\mathbf{u}_t)|^2$$
$$+ \frac{1}{2}(\chi(t) + \delta)|\varepsilon(\mathbf{u}(t))|^2 + \frac{1}{p}|\nabla\chi(t)|^p + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\le \int_\Omega w(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(s)|^2 \,\mathrm{d}x + \frac{1}{2}(\chi(s) + \delta)|\varepsilon(\mathbf{u}(s))|^2 + \frac{1}{p}|\nabla\chi(s)|^p$$
$$+ \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \,\mathrm{d}x + \int_s^t \int_\Omega g \,\mathrm{d}x$$

Generalized principle of virtual powers vs classical phase inclusion

- Any *weak solution* $(w, \mathbf{u}, \chi)$ fulfills the total energy inequality for all $t \in (0, T]$, for $s = 0$, and for almost all $0 < s \leq t$

$$\int_\Omega w(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \int_s^t (\chi + \delta)|\varepsilon(\mathbf{u}_t)|^2 + \frac{1}{2}(\chi(t) + \delta)|\varepsilon(\mathbf{u}(t))|^2 + \frac{1}{p}|\nabla\chi(t)|^p + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \int_\Omega w(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\mathbf{u}_t(s)|^2\,\mathrm{d}x + \frac{1}{2}(\chi(s) + \delta)|\varepsilon(\mathbf{u}(s))|^2 + \frac{1}{p}|\nabla\chi(s)|^p + \int_\Omega W(\chi(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x + \int_s^t \int_\Omega g\,\mathrm{d}x$$

- If $(w, \mathbf{u}, \chi)$ are "more regular" and satisfy the notion of *weak solution*, then, differentiating the energy inequality and using the chain rule, we conclude that $(w, \mathbf{u}, \chi, \xi)$ comply with

$$\left\langle \chi_t(t) - \Delta_p\chi(t) + \xi(t) + \gamma(\chi(t)) + \frac{|\varepsilon(\mathbf{u})|^2}{2} - \Theta(w(t)), \chi_t(t) \right\rangle_{W^{1,p}(\Omega)} \leq 0 \text{ for a.e.} t$$

Using the *one-sided* inequality we obtain the classical phase inclusion:

$$\exists \zeta \in L^2(0, T; L^2(\Omega)) \text{ with } \zeta(x, t) \in \partial I_{(-\infty,0]}(\chi_t(x, t)) \text{ a.e. s.t.}$$
$$\chi_t + \zeta - \Delta_p\chi + \xi + \gamma(\chi) = -\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \text{ a.e.}$$

The isothermal case: uniqueness

Let $\rho \in \mathbb{R}$. In addition to the previous hypotheses, assume that

the function *a* is constant

Then, the isothermal reversible system admits a unique solution $(\mathbf{u}, \chi)$ which continuously depends on the data

The isothermal case: uniqueness

Let $\rho \in \mathbb{R}$. In addition to the previous hypotheses, assume that

the function a is constant

Then, the isothermal reversible system admits a unique solution $(\mathbf{u}, \chi)$ which continuously depends on the data

Uniqueness of solutions for the irreversible system, even in the isothermal case, is still an open problem. This is mainly due to the triply nonlinear character of the χ equation.

The techniques used in the proof

The techniques used in the proof

- We pass to the limit in a carefully designed time-discretization scheme

The techniques used in the proof

- We pass to the limit in a carefully designed time-discretization scheme
- A key role is played by
  - the presence of the p-Laplacian with $p > d$ $\Longrightarrow$ an estimate for χ in $L^\infty(0, T; W^{1,p}(\Omega))$ $\Longrightarrow$ a suitable regularity estimate on the displacement variable $\mathbf{u}$ $\Longrightarrow$ a global-in-time bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

The techniques used in the proof

- We pass to the limit in a carefully designed time-discretization scheme
- A key role is played by
  - the presence of the p-Laplacian with $p > d \Longrightarrow$ an estimate for χ in $L^\infty(0,T;W^{1,p}(\Omega)) \Longrightarrow$ a suitable regularity estimate on the displacement variable $\mathbf{u}$ $\Longrightarrow$ a global-in-time bound on the quadratic nonlinearity $|\varepsilon(\mathbf{u})|^2$ on the right-hand side of

$$\chi_t + \mu\partial I_{(-\infty,0]}(\chi_t) - \Delta_p\chi + W'(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \vartheta$$

  - the BOCCARDO-GALLOUËT-type estimates combined with the Gagliardo-Nirenberg inequality applied to the enthalpy equation in order to obtain an $L^r(0,T;W^{1,r}(\Omega))$-estimate on the enthalpy w

The degenerating case

Hypotheses

Consider the irreversible case with the $s-$Laplacian (the previous results still hold true in this case), $\rho = 0$, and $a(\chi) = \chi$, $b(\chi) = \chi + \delta$:

Hypotheses

Consider the irreversible case with the $s-$Laplacian (the previous results still hold true in this case), $\rho = 0$, and $a(\chi) = \chi$, $b(\chi) = \chi + \delta$:

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x$$
$$+ \int_0^t \int_\Omega K(w)\nabla w \nabla \varphi \,\mathrm{d}x = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\,\mathrm{d}x\,,$$
$$\mathbf{u}_{tt} - \mathrm{div}\,((\chi+\delta)\varepsilon(\mathbf{u}_t) + (\chi+\delta)\varepsilon(\mathbf{u})) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion (in $W^{1,p}(\Omega)^*$ and a.e. in $(0,T)$)

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s(\chi) + \partial I_{[0,+\infty)}(\chi) + \gamma(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

Hypotheses

Consider the irreversible case with the $s-$Laplacian (the previous results still hold true in this case), $\rho = 0$, and $a(\chi) = \chi$, $b(\chi) = \chi + \delta$:

$$\int_\Omega \varphi(t)\, w(t)(\mathrm{d}x) - \int_0^t \int_\Omega w\varphi_t \,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \Theta(w)\varphi \,\mathrm{d}x$$

$$+ \int_0^t \int_\Omega K(w)\nabla w \nabla \varphi \,\mathrm{d}x = \int_0^t \int_\Omega g\varphi + \int_\Omega w_0 \varphi(0)\,\mathrm{d}x\,,$$

$$\mathbf{u}_{tt} - \operatorname{div}\left((\chi+\delta)\varepsilon(\mathbf{u}_t) + (\chi+\delta)\varepsilon(\mathbf{u})\right) = \mathbf{f} \text{ in } H^{-1}(\Omega;\mathbb{R}^d) \text{ a.e. in } (0,T)$$

and the subdifferential inclusion (in $W^{1,p}(\Omega)^*$ and a.e. in $(0,T)$)

$$\chi_t + \partial I_{(-\infty,0]}(\chi_t) + A_s(\chi) + \partial I_{[0,+\infty)}(\chi) + \gamma(\chi) \ni -b'(\chi)\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w)$$

where

$$A_s : H^s(\Omega) \to H^s(\Omega)^* \quad \text{with } s > \frac{d}{2}, \quad \langle A_s \chi, w\rangle_{H^s(\Omega)} := a_s(\chi, w) \text{ and}$$

$$a_s(z_1, z_2) := \int_\Omega \int_\Omega \frac{(\nabla z_1(x) - \nabla z_1(y)) \cdot (\nabla z_2(x) - \nabla z_2(y))}{|x-y|^{d+2(s-1)}}\,\mathrm{d}x\,\mathrm{d}y$$

Note that all the previous results for the non-degenerating case hold true with A_s instead of Δ_p

The total energy inequality in the degenerating case $\delta \searrow 0$

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \operatorname{div}((\chi + \delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \operatorname{div}((\chi + \delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta)$:

$$\partial_t^2 \mathbf{u}_\delta - \operatorname{div}(\sqrt{\chi + \delta}\,\boldsymbol{\mu}_\delta) - \operatorname{div}(\sqrt{\chi + \delta}\,\boldsymbol{\eta}_\delta) = \mathbf{f}$$

The total energy inequality in the degenerating case $\delta \searrow 0$

Rewrite the momentum equation

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}((\chi + \delta)\varepsilon(\partial_t \mathbf{u}_\delta)) - \mathrm{div}((\chi + \delta)\varepsilon(\mathbf{u}_\delta)) = \mathbf{f}$$

using the new variables (*quasi-stresses*) $\boldsymbol{\mu}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\partial_t \mathbf{u}_\delta)$, and $\boldsymbol{\eta}_\delta := \sqrt{\chi_\delta + \delta}\,\varepsilon(\mathbf{u}_\delta)$:

$$\partial_t^2 \mathbf{u}_\delta - \mathrm{div}(\sqrt{\chi + \delta}\,\boldsymbol{\mu}_\delta) - \mathrm{div}(\sqrt{\chi + \delta}\,\boldsymbol{\eta}_\delta) = \mathbf{f}$$

The total energy inequality for $(w_\delta, \mathbf{u}_\delta, \chi_\delta)$ is

$$\begin{aligned}
&\int_\Omega w_\delta(t)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(t)|^2\,\mathrm{d}x + \int_s^t \int_\Omega |\partial_t \chi_\delta|^2\,\mathrm{d}x + \frac{1}{2}\int_s^t |\boldsymbol{\mu}_\delta(r)|^2 \\
&\quad + \frac{|\boldsymbol{\eta}_\delta(t)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(t), \chi_\delta(t)) + \int_\Omega W(\chi_\delta(t))\,\mathrm{d}x \\
&\le \int_\Omega w_\delta(s)(\mathrm{d}x) + \frac{1}{2}\int_\Omega |\partial_t \mathbf{u}_\delta(s)|^2\,\mathrm{d}x + \frac{|\boldsymbol{\eta}_\delta(s)|^2}{2} + \frac{1}{2} a_s(\chi_\delta(s), \chi_\delta(s)) \\
&\quad + \int_\Omega W(\chi_\delta(s))\,\mathrm{d}x + \int_s^t \int_\Omega \mathbf{f} \cdot \partial_t \mathbf{u}_\delta\,\mathrm{d}x + \int_s^t \int_\Omega g\,\mathrm{d}x
\end{aligned}$$

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 3] ($\delta = 0$) Under the previous assumptions, there exist

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)),\ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega)),\ \boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)),$$

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

such that

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 3] ($\delta = 0$) Under the previous assumptions, there exist

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)),\ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega)),\ \boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)),$$

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t),\ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}),$$

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 3] **($\delta = 0$)** Under the previous assumptions, there exist

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)),\ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega)),\ \boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)),$$

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t),\ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}),$$

the weak enthalpy equation and the *weak* momentum and phase relations

$$\partial_t^2 \mathbf{u} - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\mu}) - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\eta})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T),$$

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma(\chi))\,\varphi\,\mathrm{d}x + \int_0^T a_s(\chi,\varphi) \leq \int_0^T \int_\Omega \left(-\frac{1}{2\chi}|\boldsymbol{\eta}|^2 + \Theta(w)\right)\varphi\,\mathrm{d}x$$

$$\text{for all } \varphi \in L^2(0,T;W_+^{s,2}(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\},$$

The degenerate problem ($\delta = 0$): the existence theorem [E.R., R. Rossi, preprint arXiv:1205.3578v1 (2012)]

[Theorem 3] ($\delta = 0$) Under the previous assumptions, there exist

$$\mathbf{u} \in W^{1,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^{-1}(\Omega)),\ \boldsymbol{\mu} \in L^2(0,T;L^2(\Omega)),\ \boldsymbol{\eta} \in L^\infty(0,T;L^2(\Omega)),$$

$$w \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^1(\Omega)) \cap \mathrm{BV}([0,T];W^{1,r'}(\Omega)^*)$$

$$\chi \in L^\infty(0,T;H^s(\Omega)) \cap H^1(0,T;L^2(\Omega)), \quad \chi(x,t) \geq 0, \quad \chi_t(x,t) \leq 0 \text{ a.e.}$$

such that it holds true (a.e. in any open set $A \subset \Omega \times (0,T)$: $\chi > 0$ a.e. in A)

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t),\ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}),$$

the weak enthalpy equation and the *weak* momentum and phase relations

$$\partial_t^2 \mathbf{u} - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\mu}) - \mathrm{div}(\sqrt{\chi}\,\boldsymbol{\eta})) = \mathbf{f} \quad \text{in } H^{-1}(\Omega;\mathbb{R}^d), \text{ a.e. in } (0,T),$$

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma(\chi))\,\varphi\,\mathrm{d}x + \int_0^T a_s(\chi,\varphi) \leq \int_0^T \int_\Omega \left(-\frac{1}{2\chi}|\boldsymbol{\eta}|^2 + \Theta(w)\right)\varphi\,\mathrm{d}x$$

for all $\varphi \in L^2(0,T;W^{s,2}_+(\Omega)) \cap L^\infty(Q)$ with $\mathrm{supp}(\varphi) \subset \{\chi > 0\}$,

together with the **total energy inequality** (for almost all $t \in (0,T]$)

$$\int_\Omega w(t)(\mathrm{d}x) + \int_0^t \int_\Omega |\chi_t|^2\,\mathrm{d}x + \frac{1}{2}\int_0^t |\boldsymbol{\mu}(r)|^2 + \int_\Omega W(\chi(t))\,\mathrm{d}x + \mathcal{J}(t) = \int_\Omega w_0\,\mathrm{d}x$$

$$+\frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,\mathrm{d}x + \frac{1}{2}\chi_0|\varepsilon(\mathbf{u}_0)|^2 + \frac{1}{2}a_s(\chi_0,\chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,\mathrm{d}x\mathrm{d}r + \int_0^t \int_\Omega g\,\mathrm{d}x$$

$$\text{with } \int_0^t \mathcal{J}(r)\,\mathrm{d}r \geq \frac{1}{2}\int_0^t \left(\int_\Omega |\mathbf{u}_t(r)|^2\,\mathrm{d}x + |\boldsymbol{\eta}(r)|^2 + a_s(\chi(r),\chi(r))\right)$$

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

Suppose that the solution is more regular and $\chi > 0$ a.e.

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

Suppose that the solution is more regular and $\chi > 0$ a.e. Then the following identities hold true:

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t), \;\; \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}) \text{ a.e. in } \Omega \times (0, T)\,.$$

Hence

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

Suppose that the solution is more regular and $\chi > 0$ a.e. Then the following identities hold true:

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t), \ \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}) \text{ a.e. in } \Omega \times (0, T)\,.$$

Hence

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma(\chi))\,\varphi\,\mathrm{d}x + \int_0^T a_s(\chi, \varphi) \leq \int_0^T \int_\Omega \left(-\frac{1}{2\chi} |\boldsymbol{\eta}|^2 \varphi + \Theta(w)\varphi \right) \mathrm{d}x$$

$$\text{for all } \varphi \in L^2(0, T; W^{s,2}_+(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\},$$

coincides with

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \geq 0$$

$\forall \varphi \in L^2(0, T; H^s_-(\Omega)) \cap L^\infty(Q)$ and with $\xi \in \partial I_{[0,+\infty)}(\chi)$.

A comparison between the solution notions

Weak solution to the *degenerating* irreversible full system ($\delta = 0$) $\Longleftrightarrow$ weak solution to the *non-degenerating* irreversible full system ($\delta > 0$)

Suppose that the solution is more regular and $\chi > 0$ a.e. Then the following identities hold true:

$$\boldsymbol{\mu} = \sqrt{\chi}\,\varepsilon(\mathbf{u}_t), \;\; \boldsymbol{\eta} = \sqrt{\chi}\,\varepsilon(\mathbf{u}) \text{ a.e. in } \Omega \times (0, T)\,.$$

Hence

$$\int_0^T \int_\Omega (\partial_t \chi + \gamma(\chi))\,\varphi \,\mathrm{d}x + \int_0^T a_s(\chi, \varphi) \leq \int_0^T \int_\Omega \left(-\frac{1}{2\chi} |\boldsymbol{\eta}|^2 \varphi + \Theta(w)\varphi \right) \mathrm{d}x$$
$$\text{for all } \varphi \in L^2(0, T; W_+^{s,2}(\Omega)) \cap L^\infty(Q) \text{ with } \mathrm{supp}(\varphi) \subset \{\chi > 0\},$$

coincides with

$$\int_0^T \int_\Omega \chi_t \varphi + a_s(\chi, \varphi) + \xi\varphi + \gamma(\chi)\varphi + \frac{|\varepsilon(\mathbf{u})|^2}{2}\varphi - \Theta(w)\varphi \geq 0$$

$\forall \varphi \in L^2(0, T; H^s_-(\Omega)) \cap L^\infty(Q)$ and with $\xi \in \partial I_{[0,+\infty)}(\chi)$. Subtracting from the degenerate total energy inequality the weak enthalpy equation tested by 1, we recover (a.e. in $(0, T]$) the energy inequality:

$$\int_0^t \int_\Omega |\chi_t|^2 \,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2} a_s(\chi(t), \chi(t)) + \int_\Omega W(\chi(t))\,\mathrm{d}x$$
$$\leq \frac{1}{2} a_s(\chi_0, \chi_0) + \int_\Omega W(\chi_0)\,\mathrm{d}x + \int_0^t \int_\Omega \chi_t \left(-\frac{|\varepsilon(\mathbf{u})|^2}{2} + \Theta(w) \right) \mathrm{d}x\,\mathrm{d}r$$

Open problem: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

$$\vartheta_t + \chi_t \vartheta - \Delta\vartheta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2$$

Open problem: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

$$\vartheta_t + \chi_t\vartheta - \Delta\vartheta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2$$

Is should be possible to couple the weak equations for $\mathbf{u}$ and χ with

✓ the entropy production

$$\int_0^T \int_\Omega \Big((\log\vartheta + \chi)\,\partial_t\varphi - \nabla\log\vartheta\cdot\nabla\varphi\Big)\,dxdt \leq \int_0^T \int_\Omega \frac{1}{\vartheta}\left(-|\chi_t|^2 - \chi|\varepsilon(\mathbf{u}_t)|^2 - \nabla\log\vartheta\cdot\nabla\vartheta\right)\varphi\,dxdt$$

for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$

Open problem: an entropic formulation for the damage phenomena

We worked here with the small perturbation assumption, i.e. neglecting the quadratic contribution on the r.h.s in the internal energy balance:

$$\vartheta_t + \chi_t\vartheta - \Delta\vartheta = |\chi_t|^2 + \chi|\varepsilon(\mathbf{u}_t)|^2$$

Is should be possible to couple the weak equations for $\mathbf{u}$ and χ with

- ✓ the entropy production

$$\int_0^T\int_\Omega\Big((\log\vartheta + \chi)\,\partial_t\varphi - \nabla\log\vartheta\cdot\nabla\varphi\Big)dxdt$$
$$\leq \int_0^T\int_\Omega \frac{1}{\vartheta}\left(-|\chi_t|^2 - \chi|\varepsilon(\mathbf{u}_t)|^2 - \nabla\log\vartheta\cdot\nabla\vartheta\right)\varphi dxdt$$

  for every test function $\varphi \in \mathcal{D}(\overline{Q}_T)$, $\varphi \geq 0$ and

- ✓ the energy conservation

$$E(t) = E(0) \text{ for a.e. } t \in [0,T],$$

  where

$$E \equiv \int_\Omega\left(\vartheta + W(\chi) + \frac{1}{2}a_s(\chi,\chi) + \frac{|\mathbf{u}_t|^2}{2} + \chi\frac{|\varepsilon(\mathbf{u})|^2}{2}\right)dx\,.$$

This is still an open problem...

Thanks for your attention!

cf. http://www.mat.unimi.it/users/rocca/

Remarks

Remarks

The proof of Theorem 3 strongly relies on the following properties:

Remarks

The proof of Theorem 3 strongly relies on the following properties:

1. the compact embedding of $H^s(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;

Remarks

The proof of Theorem 3 strongly relies on the following properties:

1. the compact embedding of $H^s(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;
2. the fact that the s-Laplacian operator is linear: if instead we had stayed with the p-Laplacian operator, we would have not been able to pass to the limit in the nonlinear term $|\nabla \chi_\delta|^{p-2}\nabla\chi_\delta\nabla\zeta$ featuring in the χ-inequality in place of $a_s(\chi_\delta, \zeta)$;

Remarks

The proof of Theorem 3 strongly relies on the following properties:

1. the compact embedding of $H^s(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;
2. the fact that the s-Laplacian operator is linear: if instead we had stayed with the p-Laplacian operator, we would have not been able to pass to the limit in the nonlinear term $|\nabla \chi_\delta|^{p-2}\nabla\chi_\delta\nabla\zeta$ featuring in the χ-inequality in place of $a_s(\chi_\delta,\zeta)$;
3. the fact that $t \mapsto \chi_\delta(t,x)$ is nonincreasing for all $x \in \overline{\Omega}$, which follows from the irreversibility constraint;

Remarks

The proof of Theorem 3 strongly relies on the following properties:

1. the compact embedding of $H^s(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;
2. the fact that the s-Laplacian operator is linear: if instead we had stayed with the p-Laplacian operator, we would have not been able to pass to the limit in the nonlinear term $|\nabla\chi_\delta|^{p-2}\nabla\chi_\delta\nabla\zeta$ featuring in the χ-inequality in place of $a_s(\chi_\delta,\zeta)$;
3. the fact that $t \mapsto \chi_\delta(t,x)$ is nonincreasing for all $x \in \overline{\Omega}$, which follows from the irreversibility constraint;
4. the fact that we neglige the thermal expansion, i.e. we take $\rho = 0$, is due to the low regularity estimates we have on $\operatorname{div}\mathbf{u}_t$ for $\delta = 0$, which does not allow to pass to the limit in $\rho\operatorname{div}(\mathbf{u}_t)\Theta(w)$ when $\delta \searrow 0$

Remarks

The proof of Theorem 3 strongly relies on the following properties:

1. the compact embedding of $H^s(\Omega)$ into $\mathrm{C}^0(\overline{\Omega})$;
2. the fact that the s-Laplacian operator is linear: if instead we had stayed with the p-Laplacian operator, we would have not been able to pass to the limit in the nonlinear term $|\nabla \chi_\delta|^{p-2}\nabla\chi_\delta\nabla\zeta$ featuring in the χ-inequality in place of $a_s(\chi_\delta,\zeta)$;
3. the fact that $t \mapsto \chi_\delta(t,x)$ is nonincreasing for all $x \in \overline{\Omega}$, which follows from the irreversibility constraint;
4. the fact that we neglige the thermal expansion, i.e. we take $\rho = 0$, is due to the low regularity estimates we have on $\operatorname{div}\mathbf{u}_t$ for $\delta = 0$, which does not allow to pass to the limit in $\rho\operatorname{div}(\mathbf{u}_t)\Theta(w)$ when $\delta \searrow 0$

These are the reasons why we have restricted the analysis of **the degenerate limit** to the **irreversible system**, with the **nonlocal s-Laplacian operator**.