Advanced Mathematical Methods for Engineers

February 20, 2018 Solutions

1) The largest interval where the coefficients

1.1) of the linear ODE are continuous is $(0,+\infty)$. Moreover, the equation can be rewritten as $y' = f(x,y) := \frac{2y}{x} + 3x^b$ and f is globally Lipschitz in y in $(0,+\infty)\times\mathbb{R}$ and $f \in C^0((0,+\infty)\times\mathbb{R})$ and so we have $\exists!$ global solution $y_b \in C^1(0,+\infty)$ $\forall b>0$.

1.2)
$$y_b(x) = e^{\int_1^x \frac{2}{t}dt}\left(2+\int_1^x 3t^b e^{-\int_1^t \frac{2}{s}ds}dt\right)$$
$$= x^2\left(2+\int_1^x 3t^{b-2}dt\right)$$
$$\begin{cases} \overset{\text{if } b\neq 1}{=} 2x^2 + \dfrac{3x^{b+1}}{b-1} - \dfrac{3x^2}{b-1} \\ \\ \overset{\text{if } b=1}{=} x^2(2+3\log x) \end{cases}$$

1.3) If $b=1$ then $y_b \xrightarrow[x\to+\infty]{} +\infty$

If $b+1>2$, i.e. $b>1$, then

$y_b \xrightarrow[x\to+\infty]{} +\infty$

If $b+1<2$ we need $2-\frac{3}{b-1}>0$, i.e. $b<5/2$

Hence all values of $b>0$ are ok.

2) The general integral is

$$X(t) = e^{At}C \quad \forall C \in \mathbb{R}^3$$

Let us compute the eigenvalues of A:

$$0 = \det(A-\lambda I) = (2-\lambda)(\lambda^2+1)$$

Hence we get $\lambda_{1,2} = \pm i$, $\lambda_3 = 2$ with multiplicity 1.

Solving $(A-iI)\underline{v}=0$ we get

$$\begin{cases} (1-i)x + z = 0 \\ y = 0 \\ -2x-(1+i)z = 0 \end{cases} \Rightarrow \omega_1 = (1,0,i-1) \text{ is one eigenvector associated to } i$$

$\underline{v} = (x,y,z)$

and so $\omega_2 = \bar{\omega}_1 = (1,0,-1-i)$ is associated to $-i$.

Solving $(A-2I)\underline{v}=0$ we get $\omega_3 = (2,-5,-3)$ as one eigenvector.

Then we get a basis of solutions:

$$X_1(t) = \mathrm{Re}(e^{\lambda_1 t}\omega_1) = \begin{pmatrix} \cos t \\ 0 \\ -\cos t - \operatorname{sen} t \end{pmatrix}$$

$$X_2(t) = \mathrm{Im}(e^{\lambda_1 t}\omega_1) = \begin{pmatrix} \operatorname{sen} t \\ 0 \\ \cos t - \operatorname{sen} t \end{pmatrix}$$

$$X_3(t) = e^{\lambda_3 t}\omega_3 = \begin{pmatrix} 2e^{2t} \\ -5e^{2t} \\ -3e^{2t} \end{pmatrix}$$

And so we get

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$$

$\forall c_1, c_2, c_3 \in \mathbb{R}$.

3) a) In order to see if $f_n \in L^1(0,+\infty)$ $\forall n$ we need to study f_n in $\mathcal{U}(0+)$ and in $\mathcal{U}(+\infty)$ because on all bdd intervals inside $(0,+\infty)$ f_n are continuous and so integrable.

In $\mathcal{U}(0+)$

$$f_n(x) \sim \frac{nx}{nx^{3/2}} = \frac{1}{x^{1/2}} \in L^1(\mathcal{U}(0+)) \quad \forall n$$

In $\mathcal{U}(+\infty)$

$$|f_n(x)| \leq \left| \frac{\sin nx}{nx^{3/2}} \right| \leq \frac{1}{nx^{3/2}} \in L^1(\mathcal{U}(+\infty)) \quad \forall n$$

b) $\forall x \in (0,+\infty)$ $\lim_{n \to +\infty} f_n(x) = 0$

because $0 \leq |f_n(x)| \leq \frac{1}{nx^{3/2}} \xrightarrow{n \to +\infty} 0 \quad \forall x$

c) We can use the Lebesgue dominated convergence theorem.

For $x \in [0,1]$ we have

$$|f_n(x)| \leq \frac{nx}{nx^{3/2}} \leq \frac{1}{x^{1/2}} \in L^1(0,1)$$

For $x \geq 1$ we have

$$|f_n(x)| \leq \frac{1}{nx^{3/2}} \leq \frac{1}{x^{3/2}} \in L^1(1,+\infty)$$

$$\forall n \geq 1$$

And so we can use Lebesgue with

$$|f_n(x)| \leq g(x) = \begin{cases} \frac{1}{x^{1/2}} & x \in (0,1] \\ \frac{1}{x^{3/2}} & x > 1 \end{cases}$$

$$\underset{L^1(0,+\infty)}{\cap}$$

$$\Rightarrow \int_0^{+\infty} f_n(x)dx \longrightarrow \int_0^{+\infty} 0\, dx = 0$$

4) Let us compute $L_0(x) = 1$,

$$L_1(x) = 1 - x$$

$$L_2(x) = 1 - 2x + \frac{x^2}{2}$$

Then the projection requested is

$$Pf(x) = (f, L_0) L_0 + (f, L_1) L_1 + (f, L_2) L_2$$

$$(f, L_0) = \int_0^{+\infty} e^{x/4} e^{-x} dx = -\frac{4}{3} e^{-\frac{3x}{4}}\Big|_0^{+\infty}$$

$$= \frac{4}{3}$$

$$(f, L_1) = \int_0^{+\infty} e^{-\frac{3x}{4}}\,dx - \int_0^{+\infty} x e^{-\frac{3x}{4}}\,dx$$

$$= \frac{4}{3} + \left[\frac{4}{3} x\, e^{-\frac{3}{4}x}\right]_0^{+\infty} - \int_0^{+\infty} \frac{4}{3} e^{-\frac{3}{4}x}\,dx$$

$$= \frac{4}{3} + 0 - \frac{16}{9} = -\frac{4}{9}$$

$$(f, L_2) = \int_0^{+\infty} e^{-\frac{3x}{4}}\,dx - 2\int_0^{+\infty} x e^{-\frac{3x}{4}}\,dx$$

$$+ \frac{1}{2}\int_0^{+\infty} x^2 e^{-\frac{3x}{4}}\,dx =$$

$$= \frac{4}{3} - 2\cdot\frac{16}{9} + \frac{1}{2}\left[-\frac{4}{3} x^2 e^{-\frac{3x}{4}}\right]_0^{+\infty}$$

$$+ \frac{1}{2}\int_0^{+\infty} \frac{4}{3}\cdot 2x e^{-\frac{3x}{4}}\,dx$$

$$= \frac{4}{3} - 2\cdot\frac{16}{9} + \frac{1}{2}\cdot\frac{8}{3}\int_0^{+\infty} x e^{-\frac{3x}{4}}\,dx$$

$$= \frac{4}{3} - 2\cdot\frac{16}{9} + \frac{1}{2}\cdot\frac{8}{3}\cdot\frac{16}{9} = \frac{4}{27}$$

And so $$Pf(x) = \frac{4}{3} - \frac{4}{9}(1-x) + \frac{4}{27}\left(1-2x+\frac{x}{2}\right.$$

$$= \frac{28}{27} + \frac{4}{27}x + \frac{2}{27}x^2$$