Advanced Math. Meth. for Engineers | 21.06.'17

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Solutions

1) The equation is at separable variables.

If $a=2 \Rightarrow y \equiv 2$ is solution defined on $\mathbb{R}$

If $a \neq 2 \Rightarrow y(x) \neq 2 \ \forall x \in \mathrm{dom}(y)$

and $(y-2)^{-1} - (a-2)^{-1} = \cos x - 1$

$\Rightarrow (y-2)^{-1} = (a-2)^{-1} + \cos x - 1$

If $a > 2 \Rightarrow y > 2$ and so

$\mathrm{dom}(y) \equiv \mathbb{R}$ iff $(a-2)^{-1} + \cos x - 1 > 0$

$\Leftrightarrow \frac{1}{a-2} - 1 > 1 \Leftrightarrow 2 < a < 5/2$

If $a < 2 \Rightarrow y < 2$ and so

$\mathrm{dom}(y) \equiv \mathbb{R}$ iff $(a-2)^{-1} + \cos x - 1 < 0$
iff $a < 2$

Then $\mathrm{dom}(y) \equiv \mathbb{R}$ iff $\boxed{a < 5/2}$

2) a) The critical points of the system are:

$$\begin{cases} y = 0 \\ -x - y - (x^2 + y^2) = 0 \end{cases}$$

i.e. $\begin{cases} y = 0 \\ -x(1+x) = 0 \end{cases}$ i.e. $\begin{cases} P_1 = (0,0) \\ P_2 = (-1,0) \end{cases}$

b) Compute the Jacobian $JF(x,y)$ | 21.06.17 ②

where $F(x,y) = (y, -x-y-(x^2+y^2))$

$$JF(x,y) = \begin{pmatrix} 0 & 1 \\ -1-2x & -1-2y \end{pmatrix}$$

In P_1 we have $JF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

and the corresponding linearized system is

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x-y \end{cases}$$

The characteristic polynomial is

$\lambda^2 + \lambda + 1$ with roots $\lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2}$

and since $Re(\lambda_{1,2}) < 0$

$\Rightarrow$ P_1 is asymptotically stable.

In P_2 we have $JF(-1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

Putting $X = x-(-1)$, $Y = y$ we get the linearized system

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = X-Y \end{cases}$$

whose characteristic eq is $\lambda^2 + \lambda - 1 = 0$

and roots are $\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow$ P_2 is unstable (saddle point)

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3) $0 \le f_n(x) \le \sup_{x\in[0,+\infty)} |g_n(x)|$

$= g_n(\sqrt[4]{3n}) = \frac{3^{3/4}}{4\sqrt[4]{n}} \xrightarrow[n\to\infty]{} 0$

$g_n(x) = \frac{x^3}{n+x^4}$

$\Rightarrow f_n \to 0$ pointwise and also

$\sup_{x\in[0,+\infty)} |f_n(x) - 0| = \frac{3^{3/4}}{4\sqrt[4]{n}} \xrightarrow[n\to\infty]{} 0$

4) $u(x,t) = v(x)\,w(t)$

$w'(t) = \lambda\, w(t)$

$\Rightarrow w(t) = c e^{\lambda t} \quad c \in \mathbb{R}$

We need to find the eigenvalues/vectors of

$$\begin{cases} v''(x) - \lambda v(x) = 0 \\ v'(0) = v'(\pi) = 0 \end{cases} \quad \lambda \in \mathbb{R}$$

1) $\lambda = \mu^2 > 0 \Rightarrow v(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$+ \quad v'(0)\,w(t) = v'(\pi)\,w(t) = 0 \quad \forall t > 0$

$\Rightarrow \quad c_1 = c_2 = 0 \quad \Rightarrow v \equiv 0$

2) $\lambda = 0 \Rightarrow v(x) = c_1 + c_2 x$

and $c_2 = 0 \quad c_1 \in \mathbb{R} \Rightarrow$ constants

3) $\lambda = -\mu^2 < 0$ | 21.06.17 (4)

$v(x) = C_1 \cos \mu x + C_2 \operatorname{Sen} \mu x$

$v'(0) = v'(\pi) = 0$

$$\begin{cases} v'(x) = -\mu C_1 \operatorname{Sen} \mu x + \mu C_2 \cos \mu x \\ v'(0) = 0 \end{cases}$$

$\Rightarrow C_2 = 0$

$v'(\pi) = 0 \Rightarrow \mu = k \in \mathbb{N},\ C_2 \in \mathbb{R}$

$\Rightarrow \lambda_k = -k^2$ and $v_k(x) = \cos kx$

$$\Rightarrow u(x,t) = \sum_{k=0}^{\infty} c_k \underbrace{e^{-k^2 t} \cos kx}_{v_k(x,t)}$$

where c_k :

$$u(x,0) = \sum c_k \cos kx = f(x)$$

Prolongating f as even function on $[-\pi, \pi]$ we get

$$f(x) = \frac{f_0}{2} + \sum f_k \cos kx$$

$$f_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx \, dx$$

$\Rightarrow$ we get : $c_0 = f_0/2$

$c_k = f_k \Rightarrow$

$$u(x,t) = \frac{f_0}{2} + \sum_{k=1}^{\infty} f_k v_k(x,t)$$

$$= \frac{f_0}{2} + \sum_{k=1}^{\infty} f_k e^{-k^2 t} \cos kX$$

Uniqueness

Let u, v be two solutions and

$w := u - v$ and $E(t) = \int_0^\pi w^2(x,t)\,dx$

$\Rightarrow E(t) \geq 0 \quad E(0) = 0$

$$E'(t) = 2\int_0^\pi w w_t = 2\int_0^\pi w\, w_{xx}\,dx$$

$$= -2\int_0^\pi (w_x)^2\,dx \leq 0$$

$\Rightarrow E \downarrow$ and so $E = 0 \;\forall t$

$\Rightarrow w(x,t) \equiv 0$

$\uparrow$

w continuous