Advanced Mathematical Methods

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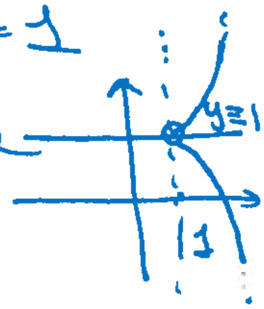
1) 1.1) On $(1,+\infty)\times\mathbb{R}$ $\quad f(x,y)=\frac{2x}{x^2-1}(y-1)$ is

$C^1 \Rightarrow \exists!$ local solutions $y:\mathrm{dom}\,y\to\mathbb{R}$

in $\mathrm{dom}\,y = \mathcal{U}(2,y_0)$.

1.2) $y\equiv 1$ is a solution in case $y_0=1$



If $y_0<1 \Rightarrow y<1 \Rightarrow$ the solution decreases

If $y_0>1 \Rightarrow y>1 \Rightarrow$ the solution increases

1.3) By separation of variables we have

$$\int \frac{dy}{y-1} = \int \frac{2x}{x^2-1}\,dx$$

$$\Rightarrow \log|y-1| = \log k\,|x^2-1|$$

$$\Rightarrow y(x) = L\,(x^2-1)+1$$

Imposing $y(2)=y_0$ we get

$$y(x)=\left(\frac{y_0-1}{3}\right)(x^2-1)+1$$

1.4) In case $y_0=3$ we get

$$y(x)=\frac{2}{3}(x^2-1)+1$$

2) $\lambda^2+2\lambda+1=(\lambda+1)^2=0$ ②

$\lambda_1=\lambda_2=-1 \implies$

$y_0(x)=c_1e^{-x}+c_2xe^{-x} \quad \forall c_1,c_2\in\mathbb{R}$

is a solution for the homogeneous equation.

A particular solution of the ODE is

$y_p(x)=Ae^x+B, \ B,A\in\mathbb{R}$

$\implies$ the solution is of the form

$y(x)=(c_1x+c_2)e^{-x}+Ae^x+B.$

$$\lim_{x\to+\infty} e^{-x}y(x)=$$

$$=\lim_{x\to+\infty}\left((c_1x+c_2)e^{-2x}+A+Be^{-x}\right)=A$$

To find A we impose:

$y_p''+2y_p'+y_p=Ae^x+2Ae^x+Ae^x$

$=4Ae^x=8e^x$

$\implies A=2$ and $\ell=2$

3) a) $f(x)=\begin{cases}1 & 0\le x<1/3\\ \sqrt{3}/2 & x=1/3\\ 0 & x>1/3\end{cases}$

f is not continuous so we do not have uniform convergence on $[0,+\infty)$

b) No because $f\notin \mathcal{C}^0([0,+\infty))$.

c) Since $0 \le \frac{\pi}{2+(3x)^n} \le \frac{\pi}{2}$

and $x \to \sin x$ is increasing on $[0, \frac{\pi}{2}]$

$\Rightarrow x \mapsto f_n(x)$ is decreasing on $[a, +\infty)$ $\forall a$

$\Rightarrow \sup_{x \in [a,+\infty)} |f_n(x) - f(x)| = \sup_{x \in [a,+\infty)} |f_n(x)| =$

$\uparrow$ if $a > 1/3$

$= f_n(a) \xrightarrow[n \to +\infty]{} 0$ if $a > 1/3$

So we have convergence in C^0 only if $a > 1/3$.

4) b) Let $\varphi \in \mathcal{D}(\mathbb{R})$ and compute

$$\lim_{\lambda \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\lambda}{x^2+\lambda^2} \varphi(x)\, dx =$$

$$\underset{\xi = \frac{x}{\lambda}}{=} \frac{1}{\pi} \lim_{\lambda \to 0^+} \int_{\mathbb{R}} \frac{1}{\xi^2+1} \varphi(\lambda \xi)\, d\xi =$$

$$\underset{\text{Lebesgue Theorem}}{=} \frac{1}{\pi} \int_{\mathbb{R}} \lim_{\lambda \to 0^+} \frac{1}{\xi^2+1} \varphi(\lambda\xi)\, d\xi =$$

$$= \frac{1}{\pi} \varphi(0) \pi = \varphi(0) = \langle \delta, \varphi \rangle$$

a) $$\int_{\mathbb{R}} f_\lambda(x)\, dx = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\lambda}{x^2+\lambda^2}\, dx$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\xi}{\xi^2+1} = \frac{1}{\pi} \arctan \xi \Big|_{-\infty}^{+\infty} = 1$$