20.2.2017 | Adv Math. (1)

1) $f \in C^{\infty}(\mathbb{R}^2) \Rightarrow \exists!$ local solution

Moreover

$$\left|\frac{\partial f}{\partial y}\right| \leq \begin{cases} 1 + \frac{|y|}{1+e^y} \leq c & \text{if } y > 0 \\ 1 + \frac{|y| e^y}{1+e^y} \leq c' & \text{if } y < 0 \end{cases}$$

$$\leq \max\{c, c'\}$$

$\Rightarrow$ f is globaly Lipschitz-continuous and so $\exists!$ global solution

Moreover $\exists \lim_{x \to -\infty} y(x) = \ell^- > 1$

because $y'(x) < 0$

because $y(x) > 0 \ \forall x \in \mathbb{R}$

$\Rightarrow \exists \ell_+ = \lim_{x \to +\infty} y(x) \in [0,1)$

Moreover $y'(x) \xrightarrow[x \to +\infty]{} -\frac{\ell^+}{1+e^{\ell_+}} = 0$

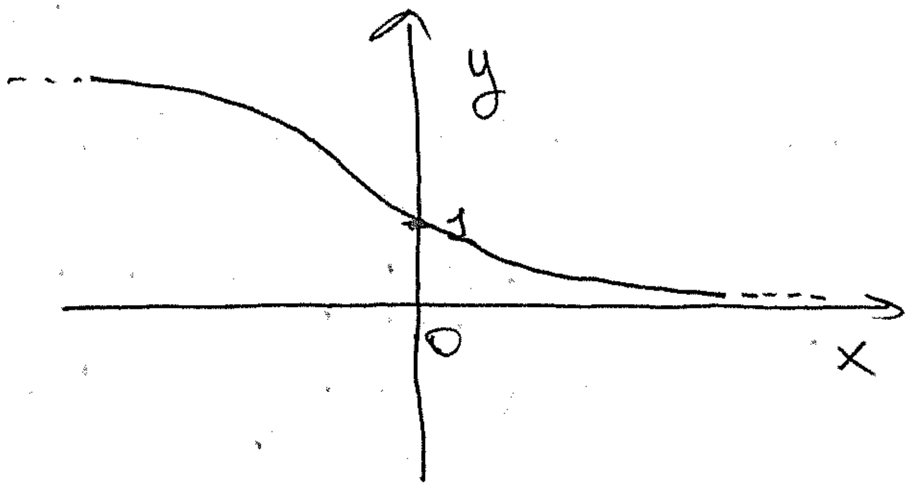
$\Rightarrow \ell^+ = 0$

and analogously $\ell^- = +\infty$

Computing $y''(x)$ it's possible to see that $y''(x) \sim -\frac{y^2(x)}{(1+e^{y(x)})^2} < 0$

when $x \to -\infty$ $(y \to +\infty)$

It's growing under a line : there are no asymphots



2) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

$\det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (\lambda-2)(\lambda+1)$

$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \underline{u} + c_2 e^{-t} \underline{v}$

relative eigenvectors

$\underline{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow$

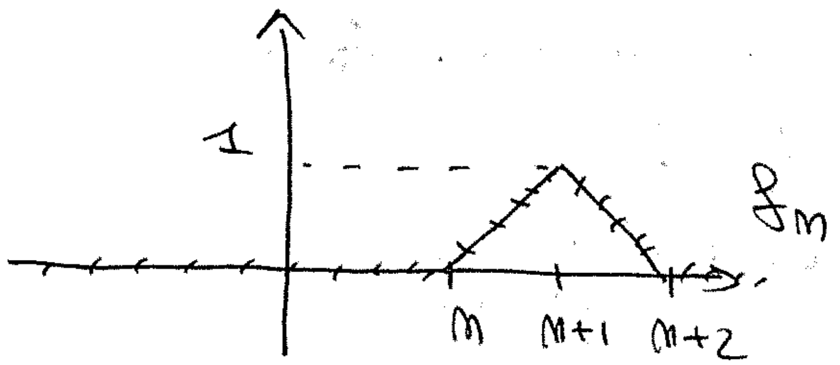
a) $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$c_1, c_2 \in \mathbb{R}$

b) The bounded solutions are the ones with $c_1 = 0 \Rightarrow$

$$\begin{cases} x(t) = c_2 e^{-t} \\ y(t) = 2c_2 e^{-t} \end{cases} \quad c_2 \in \mathbb{R}$$

4)



$'' = f_n$

$g_n \geq 0 \iff n \leq x \leq 2+n$

So if $x \geq n+1$ $f_n(x) = -x+n+2$

if $x \leq n+1$ $f_n(x) = x-n$

$\sup_{\mathbb{R}} |f_n(x)| = 1 \not\to 0$ on $\mathbb{R}$

a) $f_n \to 0$ pointwise because for $\bar{x}$ fixed and $n_0 > |\bar{x}| \Rightarrow$

$f_n(\bar{x}) = 0 \Rightarrow f_n \xrightarrow{P.} 0$

b) On compact sets

$f_n \xrightarrow{u.} 0$ because we can choose $n_0 : \forall n \geq n_0$ $\sup_{x \in [a,b]} |f_n(x)| = 0$

3) a) f_n are bounded on $[4,6]$ $\Rightarrow$

$f_n \in L^1(E)$ $\forall n$

b) $f_n(0) = -\frac{2}{n^2} \xrightarrow[n\to\infty]{} 0$

$$\underset{x\neq 0}{f_n(x)} \sim \frac{3\sqrt{n}x^2 - 24\sqrt{n}x}{n^2} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$

$f_n \xrightarrow[p]{} 0 = f(x)$ $\forall x \in E$

c) $$f_n'(x) = \frac{3x^2 + 6(\sqrt{n}-2)x - 24\sqrt{n}}{n^2} \geq 0$$

for $x \leq -2\sqrt{n}$ or $x \geq 4$

$$\Rightarrow \sup_{x\in[4,6]} |f_n(x)| = f_n(6) \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow f_n \xrightarrow[u.]{} f$ on $[4,6]$

$\Rightarrow f_n \to f$ in $L^1(E)$

d) $$\lim_{n\to\infty} \int_4^6 f_n(x)\,dx = 0$$

because $0 \leq \left| \int_4^6 f_n(x)\,dx \right| \leq f_n(6)\cdot 2 \xrightarrow[n\to\infty]{} 0$