A dual formulation for a class of phase-field systems: existence and long-time behaviour of solutions

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joint work with E. Bonetti (Pavia) and M. Frémond (Paris)

Plan of the Talk

We discuss here a new approach to phase transitions with thermal memory which consists in writing the first principle of thermodynamics in a dual formulation in the sense of Convex Analysis (cf. [B. Stinner's PhD Thesis, '05] for a similar approach).

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- Finally, we will state the long-time behaviour results for some specific cases

  E. Bonetti, E.R., Commun. Pure Appl. Anal., to appear

The internal energy: a dual formulation

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- σ and λ are smooth functions accounting for the non-convex part and the latent heat in E_P
- $\widehat{\beta} : \mathbb{R} \to (-\infty, \infty]$ is a general proper convex and lower-semicontinuous function
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to the following general free energy functional:

$$\Psi_P(\vartheta, \chi, \nabla\chi) = -\widehat{\alpha}^*(\vartheta) - \lambda(\chi)\vartheta + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

- $\widehat{\alpha}^* : \mathbb{R} \to \mathbb{R}$ is the convex conjugate of $\widehat{\alpha}$

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to the standard one in case $\boxed{\widehat{\alpha}^*(\vartheta) = c_v\vartheta(\log\vartheta - 1)}$:

$$\Psi_P(\vartheta, \chi, \nabla\chi) = c_v\vartheta(1 - \log\vartheta) - \lambda(\chi)\vartheta + \sigma(\chi) + \widehat{\beta}(\chi) + \frac{\nu}{2}|\nabla\chi|^2$$

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Referring to the paper of 1968 by Gurtin and Pipkin, in our dual formulation, we consider as state variable the summed past history of $\nabla\vartheta$ $(= \nabla(\partial E/\partial s))$ up to time t:

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- $h:\ (0,+\infty)\to(0,+\infty)$ denotes a continuous, decreasing function such that $\int_0^{+\infty}\tau^2\, h(\tau)\, d\tau < \infty$.

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- Since, c_v in the applications may also not be constant, we can allow every form for $c_v = c_v(\vartheta)$ such that $\widehat{\alpha}^*(\vartheta)$ is convex - e.g., if $c_V(\vartheta) = \vartheta^\sigma$, for $\vartheta \in (0, \bar{\vartheta})$ with $\sigma \geq 0$ - since $c_v(\vartheta) = -\vartheta\left(\partial^2\Psi/\partial\vartheta^2\right)$, then we have $\widehat{\alpha}^*(\vartheta) = \vartheta^{\sigma+1}/\left[\sigma(\sigma+1)\right]$

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  $$\widehat{\alpha}(u) = u^{\frac{\sigma+1}{\sigma}}/(\sigma+1)$$

The phase inclusion

Using the generalized principle of virtual power (cf. [Frémond, '02]), we get

$$\chi_t - \mathrm{div}\left[\frac{\partial E_P}{\partial(\nabla\chi)}\right] + \frac{\partial E_P}{\partial\chi} = 0$$

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$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \alpha(s-\lambda(\chi))\lambda'(\chi) \ni 0 \quad \text{in } \Omega$$

and $\partial_{\mathbf{n}}\chi = 0$ on $\partial\Omega$

- $\alpha = \widehat{\alpha}'$ and $\beta = \partial\widehat{\beta}$.

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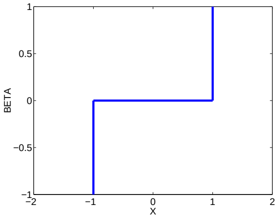
Subdifferential case: $\beta := \partial\widehat{\beta} = \partial I_{[-1,1]}$:



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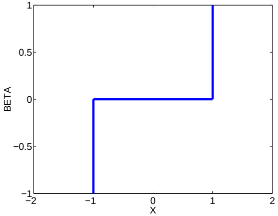
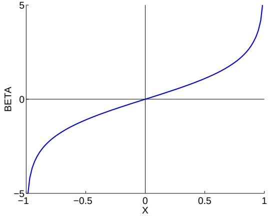


Logarithmic case: $\beta := \partial\widehat{\beta} = \log(1+\chi) - \log(1-\chi)$:



The energy balance

The first principle of thermodynamics reads

$$E_t + \mathrm{div}\,\mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial E}{\partial(\nabla\chi)}\nabla\chi_t + \chi_t^2 \quad \text{in } \Omega.$$

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and, denoting by $u = s - \lambda(\chi)$, it gives:

$$\alpha(u)\,(s_t + \operatorname{div} \mathbf{Q}) = r^{int} + \alpha'(u)|\nabla u|^2 + \chi_t^2 \text{ in } \Omega$$

where we have chosen

- $\mathbf{Q} = \dfrac{\mathbf{q}}{\alpha(u)} = -\displaystyle\int_0^{+\infty} h(\tau)\widetilde{\nabla s}^t(\tau)\,d\tau - \kappa\nabla u,$
- $\alpha = \widehat{\alpha}' = \dfrac{\partial E}{\partial s}$, $r^{int} = \dfrac{1}{2}\displaystyle\int_0^{+\infty} h(\tau)\frac{d}{d\tau}\left|\widetilde{\nabla s}^t(\tau)\right|^2 d\tau.$

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Divide by $\alpha(u) > 0$ the internal energy balance, getting

$$s_t + \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) = s_t + \operatorname{div}\mathbf{Q} \geq 0,$$

that is just the pointwise Clausius-Duhem inequality.

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We generalize now the system to the case: $\alpha = \partial\widehat{\alpha}$ MAXIMAL MONOTONE GRAPH (maybe also multivalued). Take the auxiliary variable $\boxed{u = s - \lambda(\chi)}$ and suppose to know the history term: $\operatorname{div} \int_{-\infty}^{0} k(t-\tau)\nabla\alpha(u(\tau))\,d\tau$ (we put it on the right hand side).

The PDE system

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$$
\begin{aligned}
&(u+\lambda(\chi))_t - \Delta(u + k*\alpha(u)) \ni r \quad \text{in } \Omega\\
&\partial_{\mathbf{n}}(u + k*\alpha(u)) \ni h \quad \text{on } \partial\Omega\\
&\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\alpha(u) \ni 0 \quad \text{in } \Omega\\
&\partial_{\mathbf{n}}\chi = 0 \quad \text{on } \partial\Omega.
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We must suppose from now on λ' constant ($= 1$ for simplicity).

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- An existence (of weak solutions) result under general assumptions on the nonlinearity α for a graph β with domain the whole $\mathbb{R}$ and with at most a polynomial growth at ∞

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- $\Omega \subset \mathbb{R}^3$ bdd connected domain with sufficiently smooth boundary $\Gamma := \partial\Omega$
- $t \in [0, \infty]$, $Q_t := \Omega \times (0, t)$, $\Sigma_t := \Gamma \times (0, t)$,
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Suppose moreover that

$\beta = \partial\widehat{\beta}$, $\alpha = \partial\widehat{\alpha}$, with $\widehat{\beta}$, $\widehat{\alpha} : \mathbb{R} \to (-\infty, +\infty]$ are proper, convex, and lower semicontinuous

$$\sigma \in C^2(D(\beta)), \quad \sigma'' \in L^\infty(D(\beta)), \quad \nu \geq 0$$

$$k \in W^{2,1}(0, t), \quad k(0) \geq 0, \quad k \equiv 0 \text{ if } k(0) = 0,$$

$$r \in L^2(Q_t) \cap L^1(0, T; L^\infty(\Omega)), \quad h \in L^\infty(\Sigma_t),$$

$$\langle R(t), v \rangle = \int_\Omega r(\cdot, t)v + \int_\Gamma h(\cdot, v)v_{|_\Gamma} \quad \forall v \in V$$

$$\widehat{\alpha}(u_0) \in L^1(\Omega),\ u_0 \in H,\ \chi_0 \in H,\ \nu\chi_0 \in V,\ \widehat{\beta}(\chi_0) \in L^1(\Omega).$$

Existence result for a general α

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Let T be a positive final time, HYPOTHESIS 1 be satisfied with $t = T$, and suppose moreover that $\nu > 0$, $k(0) > 0$, and there exists $p < 5$ such that

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then there exists at least a couple (u, χ) with the regularity properties

$$u \in H^1(0,T;V') \cap L^2(0,T;V),\ \chi \in H^1(0,T;H) \cap L^\infty(0,T;V),$$
$$\alpha_{V',V}(u) \in L^2(0,T;V'),$$
$$1 * \alpha_{V',V}(u) \in L^2(0,T;V) \cap C^0(0,T;H)$$

solving, a.e. in $(0,T)$, the PDE system:

$$\partial_t(u+\chi) + Au + A(k * \alpha_{V',V}(u)) \ni R, \quad \text{in } V', \qquad (1)$$
$$\partial_t\chi + \nu A(\chi) + \beta(\chi) + \sigma'(\chi) - \alpha_{V',V}(u) \ni 0 \quad \text{in } V', \qquad (2)$$
$$u(0) = u_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega.$$

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- $\alpha(u) = -1/u$: we recover, e.g., the Penrose-Fife system

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$$u \in C^0([0,T];H) \cap L^2(0,T;V), \quad \xi \in L^2(Q_T),$$

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THE PROOF IS A SUITABLE ADAPTATION OF THE ONE OF [BONETTI, COLLI, FRÉMOND, 2003] HOLDING TRUE IN CASE $\beta = \partial I_{[0,1]}$, $\sigma' = \vartheta_c$

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(iii) Perform uniform (w.r.t. ε) estimates, identifying α and β at the limit, by means of the assumptions on β listed before

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This estimate gives $|\nabla(1 * \vartheta)|_{L^\infty(0,T;H)}$, $|\chi|_{H^1(0,T;H)\cap L^\infty(0,T;V)}$, $|\widehat{\beta}(\chi)|_{L^\infty(0,T;L^1(\Omega))} \leq c$.

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(ii) We estimate $|\beta(\chi)|_{L^\infty(0,T;L^{4/3}(\Omega))}$ by using $|\beta(s)| \le c_\beta + c'_\beta|s|^p$, $p < 5$ (cf. $\boxed{\text{(beta)}}$), the Sobolev embedding in $3D$ domains, and the previous estimate on χ in $L^\infty(0,T;V)$ (holding true since $\nu > 0$).

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(iii) Then $(1) \times u$ gives $|u|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c$ and, by comparison, $|\partial_t u|_{L^2(0,T;V')} \leq c$.

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(iv) Finally, we pass to the limit identifying β by standard compactness:

$$\chi_\varepsilon \to \chi \quad \text{strongly in } C^0([0,T];L^4(\Omega)),$$

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(v) It remains to identify α. It is sufficient to deduce (cf. [Barbu]) $\limsup_{\varepsilon\searrow 0}\int_0^t(\alpha_\varepsilon(u_\varepsilon),u_\varepsilon)\le\int_0^t\langle\alpha(u),u\rangle$. Using (2), we have

$$\lim_{\varepsilon\searrow 0}\int_0^t(\alpha_\varepsilon(u_\varepsilon),u_\varepsilon)=\lim_{\varepsilon\searrow 0}\Big[\int_0^t(\partial_t\chi_\varepsilon,u_\varepsilon)$$
$$+\nu\int_0^t(\nabla\chi_\varepsilon,\nabla u_\varepsilon)+\int_0^t(\beta(\chi_\varepsilon),u_\varepsilon)+\int_0^t(\sigma'(\chi_\varepsilon),u_\varepsilon)\Big],$$

hence we need a strong convergence of $\nabla\chi_\varepsilon$ which we get by a Cauchy argument.

Long-time behaviour for α Lipschitz

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Let HYPOTHESIS 1 hold and suppose that

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Then, the ω-limit:

$$\omega(u_0, \chi_0, \nu) := \{(u_\infty, \chi_\infty) \in H \times H, \nu\chi_\infty \in V : \exists\, t_n \to +\infty, \\ (u(t_n), \chi(t_n)) \to (u_\infty, \chi_\infty) \text{ in } V' \times (V' \cap \nu H)\}$$

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is a compact, connected subset ($\neq \emptyset$) of $V' \times (V' \cap \nu H)$ and $\forall (u_\infty,\chi_\infty) \in \omega(u_0,\chi_0,\nu)$, $\exists\, \xi_\infty \in \beta(\chi_\infty)$ such that:

$$u_\infty = \frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right),$$

$$\nu A\chi_\infty + \xi_\infty + \sigma'(\chi_\infty) = \alpha\left(\frac{1}{|\Omega|}\left(-\int_\Omega \chi_\infty + c_0 + m\right)\right),$$

where $c_0 = \int_\Omega u_0 + \int_\Omega \chi_0$, $m = \int_0^\infty \left(\int_\Omega r(s) + \int_\Gamma h(s)\right) ds$.

The case $\alpha = \exp$: an existence result

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Then, there exists a quadruple $(u, \vartheta, \chi, \xi)$ such that $\vartheta \in \alpha(u)$, $\xi \in \beta(\chi)$, and

$$u \in H^1(0,T;V') \cap L^2(0,T;V), \quad \vartheta \in L^{5/3}(Q_T),$$
$$\chi \in H^1(0,T;H), \quad \nu\chi \in L^\infty(0,T;V) \cap L^{5/3}(0,T;W^{2,5/3}(\Omega)),$$
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solving system (1–2) a.e. in Q_T and the same initial conditions as before.

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- We cannot deduce that every solutions in some interval $(0, T)$ can be extended to the whole $(0, +\infty)$
- The asymptotic analysis that we are going to perform is restricted only to those solutions which are defined on $(0, +\infty)$
- In order to study the long-time behaviour of solutions let k be a strongly positive kernel and restrict ourselves to consider $\nu > 0$

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THE CASE ν, $k = 0$ has been studied in [Bonetti, in "Dissipative phase transitions" (ed. P. Colli, N. Kenmochi, J. Sprekels) (2006)]

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then we can conclude in addition that both u_∞ and χ_∞ are constants a.e. in Ω and that $(u_\infty, \chi_\infty) \in \omega(u, \chi)$ is uniquely determined by

$$u_\infty = -\chi_\infty + \frac{1}{|\Omega|}(c_0 + m),$$

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being c_0 and m defined as before. In particular, the whole trajectory $(u(t), \chi(t))$ tends to (u_∞, χ_∞) weakly in $H \times V$ and strongly in $V' \times H$ as $t \to +\infty$.

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- To study the general inclusion $\alpha(u)\,(u_t + \ell\chi_t) + \operatorname{div}\mathbf{q} \ni \chi_t^2$ without the small perturbations assumption for a *suitable nonlinear function* α and suitable choices of the heat flux $\mathbf{q}$ and of the phase dynamics.

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- The problem both for recovering uniqueness of solutions and existence of the attractor is the lack of regularity of the ϑ-component

Principle of virtual power for microscopic motion

For any subdomain $D \subset \Omega$ and any virtual microscopic velocity v,

$$P_{\text{int}}(D, \mathbf{v}) + P_{\text{ext}}(D, \mathbf{v}) = \mathbf{0},$$

where ($\mathbf{B}$ and H new interior forces)

$$P_{\text{int}}(D, \mathbf{v}) := -\int_D (Bv + \mathbf{H} \cdot \nabla \mathbf{v}),$$

$$P_{\text{ext}}(D, \mathbf{v}) := \int_D \mathbf{A} \cdot \mathbf{v} + \int_{\partial D} \mathbf{a} \cdot \mathbf{v} = 0.$$

From which (in absence of external actions) we derive an equilibrium equation in Ω

$$\boxed{B - \operatorname{div} \mathbf{H} = 0}$$

with the natural associated boundary condition on $\partial\Omega$

$$\mathbf{H} \cdot \boldsymbol{n} = 0.$$

The first Principle

For any subdomain $D \subset \Omega$ and in absence of external actions, it reads

$$\frac{d}{dt}\int_{\mathcal{D}} E \, d\Omega = -\mathcal{P}_i(\mathcal{D}, \chi_t).$$

Then, if we decide to take the following form for the power of internal actions:

$$\mathcal{P}_i(\mathcal{D}, \chi_t) = -\int_{\mathcal{D}} \left(B\chi_t + \mathbf{H}\cdot\nabla\chi_t\right) \, d\Omega,$$

and

$$B = \frac{\partial E}{\partial \chi} + \chi_t, \quad \mathbf{H} = \frac{\partial E}{\partial(\nabla\chi)},$$

we get exactly that there exists $\mathbf{q}$ such that

$$E_t + \mathrm{div}\,\mathbf{q} = \frac{\partial E}{\partial \chi}\chi_t + \frac{\partial E}{\partial(\nabla\chi)}\nabla\chi_t + \chi_t^2 \quad \text{in } \Omega.$$

Subdifferential in $V' - V$

We consider the functionals associated to $\widehat{\alpha}$

$$J_H(v) = \int_\Omega \widehat{\alpha}(v(x))dx \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \in L^1(\Omega),$$
$$J_H(v) = +\infty \quad \text{if} \quad v \in H \quad \text{and} \quad \widehat{\alpha}(v) \notin L^1(\Omega),$$
$$J_V(v) = J_H(v) \quad \text{if} \quad v \in V,$$

with their subdifferentials (cf. [Barbu])

$$\partial_{V,V'} J_V : V \to 2^{V'}, \quad \partial_H J_H : H \to 2^H.$$

Denote by $D(\partial_{V,V'} J_V) := \{v \in V : \partial_{V,V'} J_V(v) \neq \emptyset\}$ the domain of $\partial_{V,V'} J_V$. Then, for $u, \vartheta \in H$, we have (see, e.g., [Brezis])

$$\vartheta \in \partial_H J_H(u) \quad \text{if and only if } \vartheta \in \alpha(u) \quad \text{a.e. in } \Omega$$

and, thanks to the definitions of $\partial_{V,V'} J_V$ and $\partial_H J_H$, we have

$$\partial_H J_H(u) \subseteq H \cap \partial_{V,V'} J_V(u) \quad \forall u \in V.$$