Long time behaviour of a singular phase transition model

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We consider the system of equations

$$c_V\,\theta_t + \kappa\Delta\left(\frac{1}{\theta}\right) + (\lambda(\chi) + \beta\,\varphi(\chi))_t + b[\chi]\,\chi_t \;=\; 0\,,$$
$$\mu(\theta)\,\chi_t + \theta\sigma'(\chi) + \lambda'(\chi) + b[\chi] \;\in\; -(\beta+\theta)\,\partial\varphi(\chi)$$

in $\Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitzian boundary. The unknown functions are θ (the absolute temperature) and χ (the order parameter). The system is coupled with the Neumann boundary condition for $1/\theta$ and initial conditions for θ and χ.

We assume that φ is an arbitrary proper, convex, and lower semicontinuous function with values in $\mathbb{R} \cup \{+\infty\}$, (in typical models for solid-liquid phase transition, φ can be chosen for instance as the indicator function of the interval $[0,1]$), $c_V > 0$ is the constant specific heat, β is a positive constant, σ and λ are smooth bounded functions on the domain of φ describing the local dependence on χ of the entropy and of the latent heat, respectively, μ is a locally Lipschitz continuous function with quadratic growth, and b is a non-local operator of the form

$$b[\chi](x,t) \;=\; 2\int_\Omega k(x,y)\,G'(\chi(x,t)-\chi(y,t))\,dy$$

with a given smooth even function G and a bounded symmetric kernel k. The system satisfies the Claudius-Duhem inequality almost everywhere under the choice of the free energy density in the form

$$F[\theta,\chi] \;=\; c_V\theta(1-\log\theta) + \theta\sigma(\chi) + \lambda(\chi) + (\beta+\theta)\varphi(\chi) + B[\chi]$$

with

$$B[\chi](x,t) \;=\; \int_\Omega k(x,y)\,G(\chi(x,t)-\chi(y,t))\,dy\,.$$

Assuming that the initial temperature is bounded and strictly positive, we prove that the system admits a unique global solution with positive θ. If moreover $N \le 3$ and φ is bounded from below, then the temperature remains bounded from above and from below by positive constants for all times, and both $\nabla\theta$ and χ_t converge in the norm of $L^2(\Omega)$ asymptotically to 0 as $t \to \infty$. (Joint work with J. Sprekels.)