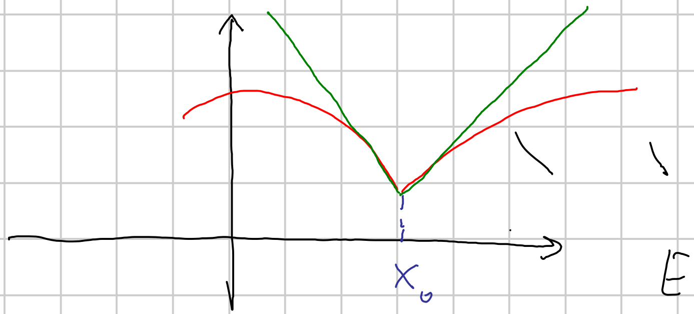
Lezione 24-10

Casi di non derivabilità:

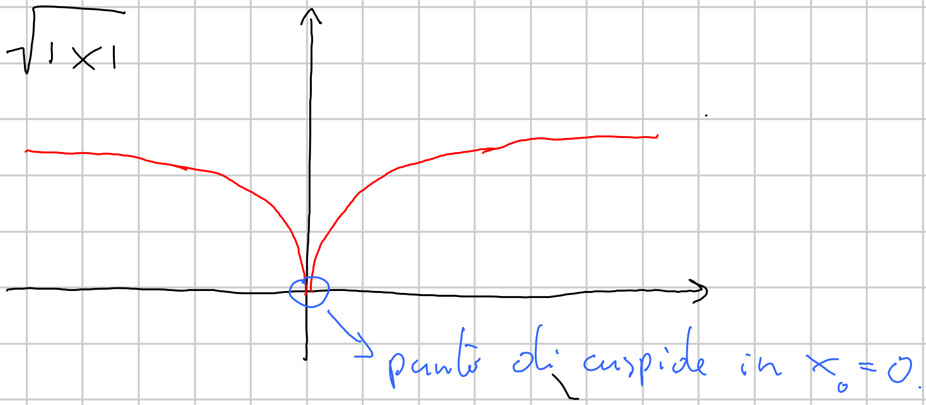
Def: $f: A \to \mathbb{R}$, $x_0 \in A$. Se esistono $f'_+(x_0)$ e $f'_-(x_0)$, entrambe finite ma $f'_+(x_0) \neq f'_-(x_0)$ allora x_0 si dice punto angoloso.



Esempio: $f(x) = |x|$.

Def: Se f è continua in x_0 e $f'_+(x_0) = +\infty$, e $f'_-(x_0) = -\infty$ (o viceversa) allora x_0 si dice punto di cuspide.

Esempio: $f(x) = \sqrt{|x|}$



Teorema: Se f e g sono derivabili in x_0 allora:

1) $f+g$ e' derivabile in x_0 e

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

2) $f \cdot g$ e' derivabile in x_0 e

$$(fg)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

3) Se $f(x_0) \neq 0 \Rightarrow \frac{1}{f}$ e' derivabile in x_0

e $\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{(f(x_0))^2}$

Oss: Combinando 2) e 3) si ottiene che se f e g sono derivabili in x_0 e $g(x_0) \neq 0$, allora f/g è derivabile in x_0 e

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$$

Infatti $\frac{f}{g} = f \cdot \frac{1}{g}$

Calcoliamo $D(e^x)$

Fissiamo $x_0 \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{e^{x_0} \cdot (e^{x - x_0} - 1)}{x - x_0} = e^{x_0} \cdot \lim_{x \to x_0} \frac{(e^{x-x_0} - 1)}{x - x_0} =$$

$t = x - x_0$

$x \to x_0 \Rightarrow t \to 0$

$$= e^{x_0} \cdot \boxed{\lim_{t \to 0} \frac{e^t - 1}{t}} = e^{x_0}$$

$$\downarrow$$

$$1$$

$\Rightarrow D(e^x) = e^x \quad \forall x \in \mathbb{R}$

- $\forall n \in \mathbb{N} \quad D^{(n)}(e^x) = e^x \quad \forall x \in \mathbb{R}.$

$D(\sin x) = \cos x$

$D(\cos x) = -\sin x$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = D(\cos x) = -\sin x$

$f'''(x) = -D(\sin x) = -\cos x$

$f^{(4)}(x) = D(-\cos x) = -D(\cos x) = \sin x = f(x)$

$f^{(5)}(x) = f^{(1)}(x)$

Per la funzione $f(x) = \sin(x)$ la derivata è ciclica di ordine 4.

Lo stesso vale per $\cos x$.

Oss: Se f e' costante $\Rightarrow f'(x) = 0 \quad \forall x$

- Se $k \in \mathbb{R} \Rightarrow D(k \cdot f) = k \cdot D(f)$

Es: $D(3 \cdot \sin x) = 3 \cdot D(\sin x) = 3 \cdot \cos x$

$$D(\operatorname{tg} x) = D\left(\frac{\sin x}{\cos x}\right) = \frac{D(\sin x) \cdot \cos x - \sin x \cdot D(\cos x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\shortparallel$$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + (tg(x))^2$$

Derivata della funzione inversa

$f : (a, b) \to \mathbb{R}$ derivabile e strettamente monotòna (quindi invertibile), allora f^{-1} è derivabile e

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \text{con } y = f(x) \quad x \in (a, b)$$

$y \in$ Im f

Es: $f(x) = e^x$, $f'(x) = e^x$. Poniamo $y = e^x$

$\Rightarrow x = \log y$

$f^{-1}(y) = \log y$

$$D(\log y) = D(f^{-1}(y)) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{e^{f^{-1}(y)}} = \frac{1}{e^{\log y}} =$$

$= \frac{1}{y}$, $D(\log y) = \frac{1}{y}$ $\forall y > 0$

$D(\log x) = \frac{1}{x}$ $\forall x > 0$.

Derivata della funzione composta:

Prop: Se f è derivabile in x_0 e g è derivabile in $f(x_0)$ allora $g \circ f$ è derivabile in x_0 e

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Es: $f(x) = \sin x$, $g(y) = e^y$

$$(g \circ f)(x) = g(f(x)) = g(\sin x) = e^{\sin x}$$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) = g'(\sin x) \cdot \cos x = e^{\sin x} \cdot \cos x$$

Es: $D(x) = 1$, $D(x^2) = D(x \cdot x) = 1 \cdot x + x \cdot 1 = 2x$

$$D(x^3) = D(x^2 \cdot x) = 2x \cdot x + x^2 \cdot 1 = 3x^2$$

$$D(x^n) = n \cdot x^{n-1} \quad \forall n \in \mathbb{N} \quad n \geq 1$$

$D(x^\alpha)$ $\quad \alpha \in \mathbb{R}, \; x > 0$

$$D(x^\alpha) = D(e^{\alpha \log x}) \overset{\text{funz. composte}}{\overset{!}{=}} e^{\alpha \log x} \cdot D(\alpha \log x) =$$

$$= e^{\alpha \log x} \alpha \cdot \frac{1}{x} = x^\alpha \cdot \alpha \cdot \frac{1}{x} = \alpha \cdot x^{\alpha - 1}$$

Es: $D(\sqrt{x}) = D(x^{\frac{1}{2}}) = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

Es: $D(a^x)$ $\quad a>0 \quad a = e^{\log a}$

$a^x = e^{x \log a}$

$D(a^x) = D(e^{x \log a}) = e^{x \log a} \cdot D(x \cdot \log a) = \log a \cdot e^{x \log a} =$

$= \log a \cdot a^x$

$D(\arctan y)$ $\qquad f(x) = \operatorname{tg} x \Rightarrow f'(x) = 1 + \operatorname{tg}^2 x$

$y = \operatorname{tg} x$

$\Rightarrow f^{-1}(y) = \arctan y$

$$\left(f^{-1}(y)\right)' = \frac{1}{f'(f^{-1}(y))} = \frac{1}{1 + \left[\operatorname{tg}(f^{-1}(y))\right]^2} = \frac{1}{1 + \left[\operatorname{tg}(\arctan y)\right]^2} =$$

$$= \frac{1}{1 + y^2}$$

Es: funzione senza derivata in un punto.

$$f(x) = \begin{cases} x \cdot \sin \dfrac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0. \end{cases} \qquad f: \mathbb{R} \to \mathbb{R}$$

f e' continua in $x_0 = 0$

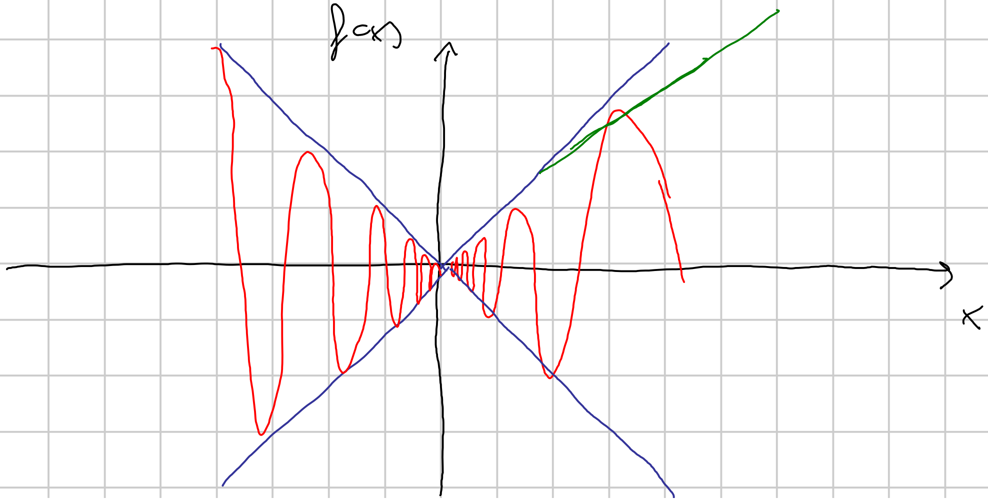
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \overset{0}{x} \cdot \underbrace{\sin \frac{1}{x}}_{\text{limitata}} = 0 = f(0)$$

Derivabilità: f è derivabile su $x \neq 0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\cancel{x}\cdot \sin\frac{1}{x}}{\cancel{x}} = \lim_{x\to 0} \sin\frac{1}{x} = \lim_{y\to\pm\infty} \sin(y)$$

non esiste

La funzione non ha derivata in $x=0$



Es: $f(x) = (1+x)^{\alpha}$ $\quad \alpha \in \mathbb{R} \quad x > -1$

$f'(x) = \alpha \cdot (1+x)^{\alpha-1}$ $\quad f(0) = 1$

$f'(0) = \alpha$

$f(x) = f(0) + f'(0) \cdot (x-0) + o(x-0) = 1 + \alpha x + o(x)$

Se $f: (a,b) \to \mathbb{R}$ è derivabile in $x_0 \in (a,b)$

allora $f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + o(x - x_0)$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \text{ , pertanto}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = 0 \text{ cioè}$$

$$\lim_{x \to x_0} \frac{\boxed{f(x) - f(x_0) - f'(x_0) \cdot (x - x_0)}}{x - x_0} = 0$$

$= o(x - x_0)$

$$f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + o(x - x_0)$$

Es: $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + o(x) \quad \left(\alpha = \frac{1}{2}\right)$

$\sqrt[3]{1+x} = (1+x)^{1/3} = 1 + \frac{1}{3}x + o(x) \quad \left(\alpha = \frac{1}{3}\right)$

Es: $\sqrt[3]{1+x^2} = 1 + \frac{1}{3}x^2 + o(x^2)$

Es: $\lim_{x \to +\infty} \left(\sqrt[3]{x^2+8x} - \sqrt[3]{x^2}\right) \cdot x^{1/3} =$

$\left[(x^2+8x)^{1/3} - x^{2/3}\right] \cdot x^{1/3} =$

$$= \left[\left(x^2\left(1+\frac{8}{x}\right)\right)^{1/3} - x^{2/3} \right] \cdot x^{1/3} =$$

$$= \left[x^{2/3} \cdot \left(1+\frac{8}{x}\right)^{1/3} - x^{2/3} \right] \cdot x^{1/3} =$$

$$= x^{2/3} \left[\left(1+\frac{8}{x}\right)^{1/3} - 1 \right] \cdot x^{1/3} = x \cdot \boxed{\left[\left(1+\frac{8}{x}\right)^{1/3} - 1 \right]} \to 0$$

$(1+t)^{\alpha} = 1+\alpha t + o(t)$ $\quad \alpha = \frac{1}{3}$

$t \to 0$

$(1+t)^{1/3} = 1 + \frac{1}{3} t + o(t)$

$t = \frac{8}{x}$ se $x \to +\infty$ $t \to 0$

Cambio di variabile

$$\cancel{x}\Big(\underbrace{\cancel{1}+\frac{1}{3}\cdot\frac{8}{\cancel{x}}+o\Big(\frac{8}{x}\Big)}_{=}-\cancel{1}\Big)=\frac{8}{3}+x\cdot o\Big(\frac{8}{x}\Big)=\frac{8}{3}$$

$$x\cdot o\Big(\frac{8}{x}\Big) \xrightarrow[x\to+\infty]{\lim} 0$$

$$x\Big(\frac{1}{3}\cdot\frac{8}{x}+\underbrace{o\Big(\frac{8}{x}\Big)}_{=f(x)}\Big)=\frac{8}{3}+\boxed{x\cdot o\Big(\frac{8}{x}\Big)}=$$

$$\boxed{x\cdot o\Big(\frac{8}{x}\Big)} \to 0$$

Se $f(x)=o\Big(\frac{8}{x}\Big)$ $\quad x\to+\infty$

$$\boxed{\lim_{x\to+\infty}\frac{f(x)}{\left(\frac{8}{x}\right)}=0}$$

$$\Updownarrow$$

$$\lim_{x\to+\infty}\Big(\frac{x}{8}\Big)\cdot f(x)=0$$

$$\Updownarrow$$

$$\lim_{x\to+\infty}x\cdot f(x)=0 \Rightarrow \lim_{x\to+\infty}x\cdot f(x)=0$$

$$\lim_{x \to +\infty} x \cdot f(x) = 0$$